

On the number of square integrable solutions and self-adjointness of symmetric first order systems of differential equations

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ABSTRACT. The main purpose of this paper is to investigate the formal deficiency indices $\mathcal{N}_{\pm}(I)$ of a symmetric first order system

$$Jf' + Bf = \lambda \mathcal{H}f$$

on an interval I , where $I = \mathbb{R}$ or $I = \mathbb{R}_{\pm}$. Here J, B, \mathcal{H} are $n \times n$ matrix valued functions and the Hamiltonian $\mathcal{H} \geq 0$ may be singular even everywhere. We obtain two results for such a system to have minimal numbers $\mathcal{N}_{\pm}(\mathbb{R}) = 0$ (resp. $\mathcal{N}_{\pm}(\mathbb{R}_{\pm}) = n$) and a criterion for their maximality $\mathcal{N}_{\pm}(\mathbb{R}_{+}) = 2n$. Some conditions for a canonical system to have intermediate numbers $\mathcal{N}_{\pm}(\mathbb{R}_{+})$ are presented, too. We also obtain a generalization of the well-known Titchmarsh–Sears theorem for second order Sturm–Liouville type equations. This contains results due to Lidskii and Krein as special cases.

We present two approaches to the above problems: one dealing with formal deficiency indices and one dealing with (ordinary) deficiency indices. Our main (non-formal) approach is based on the investigation of a symmetric linear relation S_{\min} which is naturally associated to a first order system. This approach works in the framework of extension theory and therefore we investigate in detail the domain $\mathcal{D}(S_{\min}^*)$ of S_{\min}^* . In particular, we prove the so called regularity theorem for $\mathcal{D}(S_{\min}^*)$.

The regularity result allows us to construct a bridge between the "formal" and "non-formal" approaches by establishing a connection between the formal deficiency indices \mathcal{N}_{\pm} and the usual deficiency indices $N_{\pm}(S_{\min})$. In particular we have $\mathcal{N}_{\pm} = N_{\pm}$ for definite systems.

As a byproduct of the the regularity result we obtain very short proofs of (generalizations of) the main results of the paper by Kogan and Rofo–Beketov [18] as well as a criterion for the quasi-regularity of canonical systems. This covers the Kac–Krein theorem and some results from [18].

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1. Introduction

Let $I \subset \mathbb{R}$ be an interval and consider the first order system

$$J(x)f'(x) + B(x)f(x) = \mathcal{H}(x)g(x), \quad (1.1)$$

where $J, B, \mathcal{H} : I \rightarrow M(n, \mathbb{C})$ are locally integrable matrix-valued functions with $\mathcal{H} \geq 0$ and $J(x)$ invertible (cf. (2.2) below for the precise assumptions on J, B, \mathcal{H}).

We would like to consider g in (1.1) as the result obtained by applying an operator to f . However, certain difficulties arise if $\mathcal{H} \geq 0$ is singular. It turns out that the appropriate framework to study (1.1) is the framework of symmetric linear relations in Hilbert space (Def. 2.1). To outline this let $\mathcal{L}_{\mathcal{H}}^2(I)$ be the space of \mathbb{C}^n -valued measurable functions f with $\int_I f^* \mathcal{H} f < \infty$ and denote by $L_{\mathcal{H}}^2(I)$ the corresponding Hilbert space (equivalence classes!). Then (1.1) induces symmetric linear relations, \mathcal{S}, S , in the spaces $\mathcal{L}_{\mathcal{H}}^2(I), L_{\mathcal{H}}^2(I)$ in a natural way. The first major problem which arises is the regularity problem. Suppose that one has classes $\tilde{f}, \tilde{g} \in L_{\mathcal{H}}^2(I)$ such that $\{\tilde{f}, \tilde{g}\} \in S$. Are there representatives $f, g \in \mathcal{L}_{\mathcal{H}}^2(I)$ of \tilde{f}, \tilde{g} such that (1.1) holds? In this case f would be automatically absolute continuous, because $J(x)$ is invertible. Therefore, it is appropriate to address this problem as regularity problem.

We answer this problem affirmatively (Theorem 2.4), generalizing work of Orcutt [27, Thm. II.2.6 and Thm. IV.2.5] and I. S. Kac [15], [14].

The other major purpose of this paper is to generalize several criteria for essential self-adjointness of first and second order differential operators to the present setting. We present two approaches to the above problems: one dealing with formal deficiency indices and one dealing with (ordinary) deficiency indices. Our main (non-formal) approach is based on the investigation of a symmetric linear relation S_{\min} which is naturally associated to a first order system. This approach works in the framework of extension theory and therefore we investigate in detail the domain $\mathcal{D}(S_{\min}^*)$ of S_{\min}^* . In particular, we prove the so called regularity theorem for $\mathcal{D}(S_{\min}^*)$.

More precisely, the paper is organized as follows:

In Section 2 we give a brief overview of the theory of symmetric first order systems and introduce symmetric linear relations associated with such a system. We present examples which show that on the one hand such s.l.r. may have a very exotic behavior (Example 2.2) and on the other hand that they occur quite naturally (Example 2.5). Moreover, we state the regularity Theorem (Theorem 2.4) and discuss various normal forms of symmetric first order systems using gauge transformations. For the latter we follow Kogan and Roĭe-Beketov [18].

In Subsection 2.1 we investigate the properties of \mathcal{S}, S on finite intervals. The results, in particular regularity, are summarized in Proposition 2.10. The case of an infinite interval is presented in Subsection 2.2 (Proposition 2.12).

So called *definite* systems have more pleasant properties than general systems. In Subsection 2.3 we briefly discuss such systems and present a criterion for definiteness.

Subsections 2.4 and 2.5 are devoted to defect spaces and deficiency indices. Analogously as for a symmetric operator in a Hilbert space the deficiency indices of a symmetric linear relation determine whether it is essentially self-adjoint resp. whether there exist self-adjoint extensions. In the case of the relations \mathcal{S} and S associated to a first order system one has to distinguish between the deficiency indices $N_{\pm}(S)$ of the s.l.r. S in the Hilbert space $L^2_{\mathcal{H}}(I)$ and the *formal deficiency indices* $\mathcal{N}_{\pm}(\mathcal{S})$ of the relation \mathcal{S} in the linear space $\mathcal{L}^2_{\mathcal{H}}(I)$. The latter is the dimension of *formal defect subspace* $\mathcal{E}_{\lambda} := \{f \in \mathcal{L}^2_{\mathcal{H}}(I) \mid Jf' + Bf = \lambda \mathcal{H}f\}, \lambda = \pm i$.

For arbitrary systems we establish (Proposition 2.19) the equalities $\dim \mathcal{E}_{\pm\lambda}(S) = N_{\pm}(S) + n - \text{rank } S$ which turn into the equalities $N_{\pm} = \dim \mathcal{E}_{\pm\lambda}, \lambda \in \mathbb{C}_{\pm}$, for definite systems.

This yields in particular that $\dim \mathcal{E}_{\lambda}$ is locally constant in $\mathbb{C} \setminus \mathbb{R}$ for an arbitrary (not necessarily definite) system on an arbitrary interval (Proposition 2.20). For $I = \mathbb{R}_+$ and, under more restrictive assumptions for $I = \mathbb{R}$, this fact is due to Kogan and Rofe-Beketov [18, Theorem 2.1, Theorem 2.3]. On the one hand Proposition 2.20 improves [18, Theorem 2.3] and on the other hand it gives a new proof of [18, Theorem 2.1] which is considerably simpler than the original proof. Our proof depends, however, on the regularity Theorem 2.4.

In Section 3 we discuss essential self-adjointness of the s.l.r. S on the line. The essential self-adjointness criterion Theorem 3.2 requires that \mathcal{H} is positive definite on a sufficiently large set. In Subsection 3.2 we deal with the case in which (1.1) defines a symmetric operator.

The supplementary Section 4 is included for completeness. We present an alternative proof of Theorem 3.2 using the well-known hyperbolic equation method.

Finally, Section 5 discusses in more detail the deficiency indices of the system S on the half-line. Here using simple arguments based on J. von Neumann formula we establish a connection between deficiency indices of the system S considered on the half-lines \mathbb{R}_{\pm} and on the line respectively.

Combining this formula with the regularity results from Section 2 one immediately obtains the corresponding formula for the formal deficiency indices from [18, Sec. 2.3]. Moreover, we generalize [18, Sec. 2.3] since our formula holds for arbitrary (not necessarily definite) systems. This formula allows to translate results on the half-line (about (formal) deficiency indices) into corresponding results for the line and vice versa. In particular Theorem 5.2 corresponds to Theorem 3.2. However, in Subsection 5.1 we present a proof independent of Theorem 3.2.

In Subsection 5.2 we present a criterion for essential self-adjointness in a case where the Hamiltonian \mathcal{H} is singular (Theorem 5.7). This applies in particular to second order Sturm–Liouville type equations. Our criterion generalizes result's due to Lidskii

[24] and Krein [19] and it is in the spirit of the well-known Titchmarsh–Sears theorem [6].

Furthermore, in Subsections 5.3 and 5.4 we present several other criteria which allow to determine the deficiency indices on the half line in several cases. In particular, Theorem 5.14 and Corollary 5.18 state a necessary and sufficient condition for a first order system to have maximal deficiency indices as well as to be quasiregular. These criteria have been inspired by the Kac–Krein result (see also De Brange [7]) on 2×2 canonical systems with real Hamiltonian. Our criteria cover this as well as some results on quasiregularity from [18, Sec. 3.2].

Besides, we present several examples which show the limits of the results.

Finally, in Subsection 5.5 we obtain also similar statements on quasiregularity of matrix Sturm–Liouville equation. In the scalar case these results essentially generalize Krein’s result [19] (see also [16]) mentioned above.

In conclusion we mention two recent publications [29] and [21] close to our work (see also references therein) which are devoted to self-adjointness of elliptic operators on complete manifolds.

2. The symmetric linear relation induced by a first order system

In this section we introduce the basic notation about first order systems. Denote by $M(n, \mathbb{C})$ the set of complex $n \times n$ matrices and let $I \subset \mathbb{R}$ be a (not necessarily open) interval. We denote by $AC(I)$ the set of all absolute continuous functions on I , i.e. $f \in AC(I)$ if f' exists a.e., is locally integrable, and $f(x) = \int_{x_0}^x f'(s)ds + f(x_0)$. If $U \subset \mathbb{R}^n$ is an open set, we denote by $AC(I, U)$ the set of U -valued functions whose components lie in $AC(I)$. Finally, if X is a function space over I , then X_{comp} denotes the subspace consisting of those $f \in X$ with compact support in I .

With these preparations we consider the first order system

$$J(x) \frac{df}{dx}(x) + B(x)f(x) = \mathcal{H}(x)g(x), \quad (2.1)$$

where $J, B, \mathcal{H} : I \rightarrow M(n, \mathbb{C})$ are matrix-valued functions such that:

$$\begin{aligned} J &\in AC(I, M(n, \mathbb{C})), & J(x) &= -J(x)^*, \quad \det J(x) \neq 0, \text{ for } x \in I, \\ B &\in L^1_{\text{loc}}(I, M(n, \mathbb{C})), & B(x)^* &= B(x) - J'(x), \text{ for } x \in I, \\ \mathcal{H} &\in L^1_{\text{loc}}(I, M(n, \mathbb{C})), & \mathcal{H}(x) &= \mathcal{H}(x)^*, \quad \mathcal{H}(x) \geq 0, \text{ for } x \in I. \end{aligned} \quad (2.2)$$

Let $\mathcal{L}^2_{\mathcal{H}}(I)$ be the set of Borel-measurable \mathbb{C}^n -valued functions satisfying $\langle f, f \rangle_{\mathcal{H}} := \int_I f(x)^* \mathcal{H}(x) f(x) dx < \infty$. It is well-known (cf. e.g. [1, Sec. 9], [26]) that $\mathcal{L}^2_{\mathcal{H}}(I)$ is complete with respect to the semi-norm $\|f\|_{\mathcal{H}} = \sqrt{\langle f, f \rangle_{\mathcal{H}}}$. Moreover $\mathcal{L}^2_{\mathcal{H}}(I)$ is the completion of $C_{\text{comp}}(I, \mathbb{C}^n)$ with respect to $\|\cdot\|_{\mathcal{H}}$.

We equip $\mathcal{L}^2_{\mathcal{H}}(I)$ with the (semi-definite) scalar product

$$\langle f, g \rangle_{\mathcal{H}} := \int_I f(x)^* \mathcal{H}(x) g(x) dx, \quad (2.3)$$

and put

$$L_{\mathcal{H}}^2(I) := \mathcal{L}_{\mathcal{H}}^2(I) / \{f \in \mathcal{L}_{\mathcal{H}}^2(I) \mid \|f\|_{\mathcal{H}} = 0\}. \quad (2.4)$$

$L_{\mathcal{H}}^2(I)$ is a Hilbert space. For a function $f \in \mathcal{L}_{\mathcal{H}}^2(I)$ we will denote by \tilde{f} the corresponding class in $L_{\mathcal{H}}^2(I)$. If $\mathcal{H}(x)$ is invertible a.e. then a class \tilde{f} contains at most one continuous representative, hence if $\mathcal{H}(x)$ is invertible a.e. and f is continuous then we will not distinguish between f and \tilde{f} .

If in addition $\mathcal{H}(x)$ is invertible for almost all $x \in I$ and $\mathcal{H}^{-1}, B^* \mathcal{H}^{-1} B \in L_{\text{loc}}^1(I, M(n, \mathbb{C}))$ then (2.1) induces a symmetric operator

$$L := \mathcal{H}^{-1} \left(J \frac{d}{dx} + B \right) \quad (2.5)$$

in the Hilbert space $L_{\mathcal{H}}^2(I)$ with domain $\mathcal{D}(L) = C_{\text{comp}}^1(I, \mathbb{C}^n)$ (cf. Subsection 3.2 below). The symmetry is implied by $B^* = B - J'$ and $\mathcal{H}^* = \mathcal{H}$. However, the interesting case is the one where \mathcal{H} is singular. If \mathcal{H} is singular then (2.1) will in general neither define an operator nor will it be densely defined. Rather it will give rise to *symmetric linear relations*, \mathcal{S}_{\min} resp. S_{\min} , in $\mathcal{L}_{\mathcal{H}}^2(I)$ resp. $L_{\mathcal{H}}^2(I)$ as follows: $\{f, g\} \in \mathcal{S}_{\min}$ if and only if $f \in \text{AC}_{\text{comp}}(I, \mathbb{C}^n)$, $g \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$ and $Jf' + Bf = \mathcal{H}g$.

For the reader's convenience let us briefly recall the definition of a symmetric linear relation:

DEFINITION 2.1. Let \mathfrak{H} be a linear space equipped with a positive semi-definite hermitian sesqui-linear form $\langle \cdot, \cdot \rangle$. A linear subspace $\mathcal{S} \subset \mathfrak{H} \times \mathfrak{H}$ is called a symmetric linear relation (s.l.r.) if for $\{f_j, g_j\} \in \mathcal{S}$, $j = 1, 2$, one has $\langle f_1, g_2 \rangle = \langle f_2, g_1 \rangle$.

For a s.l.r. \mathcal{S} one defines, as usual, the *domain* $\mathcal{D}(\mathcal{S}) := \{f \in \mathfrak{H} \mid \exists g \in \mathfrak{H} \{f, g\} \in \mathcal{S}\}$, the *range* $\text{im } \mathcal{S} := \{g \in \mathfrak{H} \mid \exists f \in \mathfrak{H} \{f, g\} \in \mathcal{S}\}$, and the *kernel* $\ker \mathcal{S} := \{f \in \mathfrak{H} \mid \{f, 0\} \in \mathcal{S}\}$. Furthermore, the *indeterminant* part of \mathcal{S} is defined by $\mathcal{S}(0) := \{g \in \mathfrak{H} \mid \{0, g\} \in \mathcal{S}\} = \ker(\mathcal{S}^{-1})$.

Finally, the *adjoint* of \mathcal{S} is $\mathcal{S}^* := \{\{f, g\} \in \mathfrak{H} \times \mathfrak{H} \mid \forall_{\{\phi, \psi\} \in \mathcal{S}} \langle f, \psi \rangle = \langle g, \phi \rangle\}$.

For example, the graph of an (unbounded) symmetric operator in a Hilbert space \mathfrak{H} is a s.l.r.

\mathcal{S}_{\min} induces a symmetric linear relation, S_{\min} , in $L_{\mathcal{H}}^2(I)$ in a fairly straightforward way: $\{\tilde{f}, \tilde{g}\} \in S_{\min}$ if and only if there exist representatives $f \in \tilde{f}, g \in \tilde{g}$ such that $\{f, g\} \in \mathcal{S}_{\min}$. Symmetric linear relations arising in this way have been studied thoroughly in [27]. Unfortunately, [27] has not been published and therefore is not widely available. The authors received a copy of [27] only after the present work had been almost completed. We emphasize, however, that there is only a small overlap between [27] and the present work.

In general S_{\min} will neither be densely defined nor single valued:

EXAMPLE 2.2. $I = (0, 1)$, $B = 0$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathcal{H}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. If $\{f, g\} \in \mathcal{S}_{\min}$ then $f_2' = g_1$, $f_1' = 0$, and since f is continuous with compact support we infer $f_1 = 0$.

In view of the special form of \mathcal{H} this implies $\tilde{f} = 0$. Hence, the domain of S is $\{0\}$. Note that since $g_1 = f'_2$ we have $\int_I g_1 = 0$.

Conversely, given $\tilde{g} \in \mathcal{L}^2_{\mathcal{H}}(I)$ with $\int_I g_1 = 0$ we put $f_2(x) := \int_0^x g_1(s)ds$ and $f_1 = 0$. Then $\{f, g\} \in \mathcal{S}_{\min}$ and hence $\{0, \tilde{g}\} = \{\tilde{f}, \tilde{g}\} \in S_{\min}$. Consequently, $S_{\min} = \{0\} \times \{\tilde{g} \mid g \in \mathcal{L}^2_{\mathcal{H}, \text{comp}}(I), \int_I g_1 = 0\}$ and $S_{\min}^* = \{\{\tilde{f}, \tilde{g}\} \mid f = \text{const}, g \in \mathcal{L}^2_{\mathcal{H}}(I)\}$.

This example also shows that in general S_{\min} is not closed:

DEFINITION 2.3. We denote by S the closure of S_{\min} , i.e. the minimal closed extension, and we put $S_{\max} := S_{\min}^*$. Furthermore, we write $\{f, g\} \in \mathcal{S}_{\max}$ if $f, g \in \mathcal{L}^2_{\mathcal{H}}(I)$, f is absolutely continuous, and $Jf' + Bf = \mathcal{H}g$. Finally, let \mathcal{S} be the closure of \mathcal{S}_{\min} in \mathcal{S}_{\max} , i.e. $\{f, g\} \in \mathcal{S}$ if $\{f, g\} \in \mathcal{S}_{\max}$ and there exists a sequence $(\{f_n, g_n\})_{n \in \mathbb{N}} \subset \mathcal{S}_{\min}$ such that $\|f - f_n\|_{\mathcal{H}}, \|g - g_n\|_{\mathcal{H}} \rightarrow 0$, as $n \rightarrow \infty$. That is $\mathcal{S} = \{\{f, g\} \in \mathcal{S}_{\max} \mid \{\tilde{f}, \tilde{g}\} \in S\}$.

If $\mathcal{H}(x)$ is invertible a.e. then S will at least be a single valued symmetric operator, i.e. $\{\tilde{f}, \tilde{g}_1\}, \{\tilde{f}, \tilde{g}_2\} \in S$ implies $\tilde{g}_1 = \tilde{g}_2$. We emphasize that S may be a densely defined operator even if \mathcal{H} is singular on a subset of positive Lebesgue measure. E. g. this is the case for $I = \mathbb{R}_+$ if $\int_{\alpha}^{\beta} \mathcal{H}(t)dt$ is positive definite for all $\alpha, \beta \in [0, \infty)$, $\alpha < \beta$ (see [20]).

A complete description of the indeterminant part $S(0) = \{g \mid \{0, g\} \in S\}$ for 2×2 canonical systems has been obtained in [14], [15].

The relations \mathcal{S}, S will be addressed as the symmetric linear relation of the first order system (2.1). We will write

$$\mathcal{S}(J, B, \mathcal{H}) \quad (\text{resp. } S(J, B, \mathcal{H})) \quad (2.6)$$

if we want to emphasize the dependence on J, B, \mathcal{H} .

Next we discuss the regularity problem. In view of Definition 2.3 integration by parts shows immediately that $\{f, \tilde{g}\} \in S_{\max}$ (resp. S) if $\{f, g\} \in \mathcal{S}_{\max}$ (resp. \mathcal{S}). Denoting by $\pi : \mathcal{L}^2_{\mathcal{H}}(I) \rightarrow L^2_{\mathcal{H}}(I)$ the quotient map, this means that

$$(\pi \oplus \pi)(\mathcal{S}_{\max}) \subset S_{\max}, \quad (\pi \oplus \pi)(S_{\min}) \subset S. \quad (2.7)$$

A priori it is not clear whether equality holds. We call this the regularity Theorem.

THEOREM 2.4 (Regularity Theorem). *Let $\{\tilde{f}, \tilde{g}\} \in S_{\max}$ (resp. S). Then for each representative $g \in \tilde{g}$ there exists $f \in \tilde{f}$ such that $\{f, g\} \in \mathcal{S}_{\max}$ (resp. \mathcal{S}).*

This theorem follows from Propositions 2.10 and 2.12 below. For definite systems (cf. Def. 2.14 below) Theorem 2.4 has been proved by Orcutt [27, Thm. II.2.6 and Thm. IV.2.5]. Another proof for (not necessarily definite) 2×2 canonical systems was given by I.S. Kac [15] in the deposited elaboration of [14]. We note also that his proof is rather long and can not be extended to $n \times n$ systems.

In sum, this important regularity result for first order systems is a kind of folklore theorem but proofs are not very available in the literature. To fill this gap and to make this article self-contained we present a proof below. We emphasize that our presentation treats the most general case, i.e. we do not assume that the first order system is definite.

This is more general than [27], [14]. Also we hope that our presentation is simpler and more perspicuous.

The system (2.1) can be simplified and put into canonical form. The construction is due to Kogan and Rofe–Beketov [18, Sec. 1.3] (see also [11]). Since we will make use of it heavily and to fix some notation, let us briefly recall this construction:

A "gauge transformation" $U \in \text{AC}(I, \text{GL}(n, \mathbb{C}))$ induces a unitary map

$$\Psi_U : \mathcal{L}_{\mathcal{H}}^2(I) \rightarrow \mathcal{L}_{\mathcal{H}}^2(I), \quad f \mapsto U^{-1}f, \quad \widetilde{\mathcal{H}} := U^* \mathcal{H} U, \quad (2.8)$$

and a simple computation shows that

$$\Psi_U \mathcal{S}(J, B, \mathcal{H}) \Psi_U^* = \mathcal{S}(\widetilde{J}, \widetilde{B}, \widetilde{\mathcal{H}}), \quad (2.9)$$

where

$$\widetilde{J} = U^* J U, \quad \widetilde{B} = U^* J U' + U^* B U, \quad \widetilde{\mathcal{H}} = U^* \mathcal{H} U. \quad (2.10)$$

In a first step one chooses $U \in \text{AC}(I, \text{M}(n, \mathbb{C}))$ such that $U^* J U = J(0)$. Thus we are reduced to the case where J is a constant matrix.

In a second step pick $x_0 \in I$ and let $Y(\cdot, \lambda) : I \rightarrow \text{M}(n, \mathbb{C})$ be the solution of the initial value problem

$$JY'(x, \lambda) + B(x)Y(x, \lambda) = \lambda \mathcal{H}(x)Y(x, \lambda), \quad Y(x_0, \lambda) = I_n. \quad (2.11)$$

Here, I_n denotes the $n \times n$ unit matrix. The existence of Y follows from the fact that B and \mathcal{H} are locally integrable. For $Y(x, 0)$ we simply write $Y(x)$. If $g \in \mathcal{L}_{\mathcal{H}, \text{loc}}^2(I)$ then, since $\sqrt{\mathcal{H}} \in \mathcal{L}_{\text{loc}}^2(I, \text{M}(n, \mathbb{C}))$, we have $\mathcal{H}g \in \mathcal{L}_{\text{loc}}^1(I, \mathbb{C}^n)$. Thus, the solution of the inhomogeneous initial value problem

$$Jy'(x, \lambda) + B(x)y(x, \lambda) = \lambda \mathcal{H}(x)y(x, \lambda) + \mathcal{H}(x)g(x), \quad y(x_0, \lambda) = 0, \quad (2.12)$$

exists and is unique. Taking into account the well-known (and easy to verify) formula

$$Y(x, \bar{\lambda})^* J Y(x, \lambda) = J, \quad \lambda \in \mathbb{C}, \quad (2.13)$$

the variation of constants formula reads

$$y(x, \lambda) = (K_\lambda g)(x) = Y(x, \lambda) \int_{x_0}^x J^{-1} Y(t, \bar{\lambda})^* \mathcal{H}(t) g(t) dt. \quad (2.14)$$

As with Y we write K instead of K_0 . Now we can choose Y as the gauge transformation. In view of (2.13) and (2.10) the gauge transformation Y transforms the system into a system \widetilde{S} with

$$\widetilde{J} = J(0), \quad \widetilde{B} = 0, \quad \widetilde{\mathcal{H}} = U^* \mathcal{H} U. \quad (2.15)$$

Such systems are called "canonical" in the literature.

Another choice of gauge is possible if \mathcal{H} is absolutely continuous and invertible. Then the gauge $U = \mathcal{H}^{-1/2}$ turns the system into one with $\widetilde{\mathcal{H}} = 1$. The interesting cases, however, are those with singular \mathcal{H} .

Despite the existence of canonical forms obtained from appropriate gauges we prefer to work in the framework of (2.1) since finding the canonical system corresponding to the first order system (2.1) depends on finding the fundamental system of solutions.

Another reason for working in our framework is the following: we will give criteria for S being essentially self-adjoint below. These criteria are only sufficient and not gauge invariant, hence it is desirable to have them at hand also for first order systems which are not in canonical form. It would be nice, however, to have a necessary and sufficient characterization of essential self-adjointness. Such a criterion would necessarily have to be gauge invariant. The discovery of such a criterion, however, remains an open problem.

Some remarks are in order about why first order systems are interesting. First order systems are not as special as they seem to be. Namely, an arbitrary symmetric n^{th} -order system is *unitarily* equivalent to a symmetric first order system ([18], [27]). In most cases, however, the Hamiltonian \mathcal{H} of this first order system will be singular. Instead of reproducing this result we will present two important examples. First, we show how a second order Sturm–Liouville type (quasi-differential) equation can be transformed into a system of the form (2.1).

EXAMPLE 2.5. 1. We consider a weighted Sturm–Liouville type (quasi-differential) equation

$$-\frac{d}{dx}\left(A(x)^{-1}\frac{du}{dx}(x) + Q(x)u(x)\right) + Q(x)^*\frac{du}{dx}(x) + R(x)u(x) = \mathcal{H}(x)v(x), \quad (2.16)$$

where $A, Q, R, \mathcal{H} \in L^1_{\text{loc}}(I, M(n, \mathbb{C}))$, $A(x)$ is positive definite for all $x \in I$, and $\mathcal{H}(x) \geq 0$. The system (2.16) defines a symmetric linear relation as follows: $\{u, v\} \in \mathcal{S}_{\min}$ if and only if $u \in \text{AC}_{\text{comp}}(I, \mathbb{C}^n)$, $A^{-1}\frac{du}{dx} + Qu \in \text{AC}_{\text{comp}}(I, \mathbb{C}^n)$, $v \in \mathcal{L}^2_{\mathcal{H}, \text{comp}}(I)$ and (2.16) holds. "Quasi-differential" means that $\frac{du}{dx}$ is not necessarily absolute continuous. As for first order systems, let $S_{\min} := \{\{\tilde{u}, \tilde{v}\} \mid \{u, v\} \in \mathcal{S}_{\min}\}$.

Next we introduce the first order system

$$\tilde{J} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}' + \tilde{B} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \tilde{\mathcal{H}} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad (2.17)$$

$$\tilde{J} := \begin{pmatrix} 0 & iI_n \\ iI_n & 0 \end{pmatrix}, \quad \tilde{B} := \begin{pmatrix} R - Q^*AQ & -iQ^*A \\ iAQ & -A \end{pmatrix}, \quad \tilde{\mathcal{H}} := \begin{pmatrix} \mathcal{H} & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.18)$$

and we denote by $\widetilde{\mathcal{S}_{\min}}, \tilde{S}$ the corresponding s.l.r. in $\mathcal{L}^2_{\tilde{\mathcal{H}}}(I), L^2_{\tilde{\mathcal{H}}}(I)$.

If $\{u, v\} \in \mathcal{S}_{\min}$ then $\{(u, i(A^{-1}u' + Qu)), (v, 0)\} \in \widetilde{\mathcal{S}_{\min}}$. Conversely, if $\{(f_1, f_2), (g_1, g_2)\} \in \widetilde{\mathcal{S}_{\min}}$ then $\{f_1, g_1\} \in \mathcal{S}_{\min}$. Hence the unitary isomorphism

$$\Phi : L^2_{\mathcal{H}}(I) \longrightarrow L^2_{\tilde{\mathcal{H}}}(I), \quad \tilde{f} \mapsto (\widetilde{f, 0}) \quad (2.19)$$

implements a unitary equivalence between S_{\min} and $\widetilde{S_{\min}}$, i.e. $(\Phi \times \Phi)^* \widetilde{S_{\min}} (\Phi \times \Phi) = S_{\min}$.

Even if S_{\min} is (the graph of) a densely defined symmetric operator in the Hilbert space $L^2_{\mathcal{H}}(I)$ the Hamiltonian $\mathcal{H}(x)$ is singular everywhere.

2. Consider a general first order system $S = S(J, B, \mathcal{H})$ as in (2.1). We define the square of \mathcal{S}_{\min} resp. S_{\min} as follows:

$$\begin{aligned}\mathcal{S}_{\min}^2 &:= \{ \{f, g\} \in \mathcal{L}_{\mathcal{H}}^2(I) \times \mathcal{L}_{\mathcal{H}}^2(I) \mid \exists_{h \in \mathcal{L}_{\mathcal{H}}^2(I)} \{f, h\}, \{h, g\} \in \mathcal{S}_{\min} \}, \\ S_{\min}^2 &:= \{ \{\tilde{f}, \tilde{g}\} \in L_{\mathcal{H}}^2(I) \times L_{\mathcal{H}}^2(I) \mid \exists_{\tilde{h} \in L_{\mathcal{H}}^2(I)} \{\tilde{f}, \tilde{h}\}, \{\tilde{h}, \tilde{g}\} \in S_{\min} \}.\end{aligned}\quad (2.20)$$

The squares of \mathcal{S}, S are defined analogously. We remark first that indeed

$$\begin{aligned}S_{\min}^2 &= \{ \{\tilde{f}, \tilde{g}\} \mid \{f, g\} \in \mathcal{S}_{\min}^2 \}, \\ S^2 &= \{ \{\tilde{f}, \tilde{g}\} \mid \{f, g\} \in \mathcal{S}^2 \}.\end{aligned}\quad (2.21)$$

To see this consider $\{\tilde{f}, \tilde{g}\} \in S^2$ (resp. S_{\min}^2). By definition there exists a $\tilde{h} \in L_{\mathcal{H}}^2(I)$ such that $\{\tilde{f}, \tilde{h}\}, \{\tilde{h}, \tilde{g}\} \in S$ (resp. S_{\min}). Let $g \in \tilde{g}$. By the regularity Theorem 2.4 there exists $h \in \tilde{h}$ such that $\{h, g\} \in \mathcal{S}$ (resp. \mathcal{S}_{\min} , in this case the regularity Theorem is not needed). Again by the regularity Theorem there exists $f \in \tilde{f}$ such that $\{f, h\} \in S$ (resp. S_{\min}). Thus $\{f, g\} \in S^2$ (resp. S_{\min}^2). Conversely, if $\{f, g\} \in \mathcal{S}^2$ (resp. \mathcal{S}_{\min}^2) then it is clear that $\{\tilde{f}, \tilde{g}\} \in S^2$ (resp. \mathcal{S}^2).

Next let $\{f, g\} \in \mathcal{S}_{\min}^2$, that is there is a $h \in \mathcal{L}_{\mathcal{H}}^2(I)$ such that $\{f, h\} \in \mathcal{S}_{\min}$ and $\{h, g\} \in \mathcal{S}_{\min}$. This is equivalent to the equation

$$\begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix}' + \begin{pmatrix} 0 & B \\ B & -\mathcal{H} \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g \\ 0 \end{pmatrix} \quad (2.22)$$

with $f, h \in \text{AC}_{\text{comp}}(I, \mathbb{C}^n), g \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$. A similar argument as under 1. shows that S_{\min}^2 is unitarily equivalent to $S_{\min}(J_1, B_1, \mathcal{H}_1)$, where

$$J_1 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & B \\ B & -\mathcal{H} \end{pmatrix}, \quad \mathcal{H}_1 = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.23)$$

Actually, this system is unitarily equivalent to a system of the form (2.18). Namely, the gauge transformation

$$U := \begin{pmatrix} I_n & 0 \\ 0 & iJ^{-1} \end{pmatrix} \quad (2.24)$$

transforms the system $S_{\min}(J_1, B_1, \mathcal{H}_1)$ into $S_{\min}(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$, where

$$\tilde{J} = \begin{pmatrix} 0 & iI_n \\ iI_n & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & iB^*J^{-1} \\ iJ^{-1}B & -(J^{-1})^*\mathcal{H}J^{-1} \end{pmatrix}, \quad \tilde{\mathcal{H}} = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.25)$$

This can be checked using the formulas (2.10).

Note that (2.25) is a special case of the structure (2.18), except that the lower right corner of \tilde{B} is only positive semi-definite. This is not a surprise since heuristically \mathcal{S}_{\min}^2 can be viewed as a second order system.

For future reference and to fix some notation let us present a type of first order systems which contains the two preceding examples as special cases. Consider the system

$$J_1 f' + B_1 f = \tilde{\mathcal{H}} g, \quad (2.26)$$

where

$$J_1 = \begin{pmatrix} 0 & J^* \\ -J & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} V & B \\ B^* - J' & -A \end{pmatrix}, \quad \widetilde{\mathcal{H}} = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.27)$$

We assume that (2.26) satisfies (2.2), that is $J \in \text{AC}(I, \text{M}(n, \mathbb{C}))$, $V, B, A, \mathcal{H} \in L^1_{\text{loc}}(I, \text{M}(n, \mathbb{C}))$, $\det J(x) \neq 0$, for $x \in I$, $V = V^*$, $A = A^*$, and $\mathcal{H}(x) \geq 0$ for $x \in I$.

As in the previous example, the system (2.26) can be transformed quite explicitly onto a system $S(J_2, B_2, \widetilde{\mathcal{H}})$ with J_2 constant. We present two normal forms. The gauge transformation (2.24) transforms the system $S(J_1, B_1, \widetilde{\mathcal{H}})$ onto $S(J_2, B_2, \widetilde{\mathcal{H}})$, where

$$J_2 = \begin{pmatrix} 0 & iI_n \\ iI_n & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} V & i(B - (J^*)')(J^{-1})^* \\ -iJ^{-1}(B^* - J') & -J^{-1}A(J^{-1})^* \end{pmatrix}. \quad (2.28)$$

The gauge transformation

$$U := \begin{pmatrix} I_n & 0 \\ 0 & iI_n \end{pmatrix} \quad (2.29)$$

transforms the system $S(J_2, B_2, \widetilde{\mathcal{H}})$ onto $S(J_3, B_3, \widetilde{\mathcal{H}})$, where

$$J_3 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} V & -(B - (J^*)')(J^{-1})^* \\ -J^{-1}(B^* - J') & -J^{-1}A(J^{-1})^* \end{pmatrix}, \quad (2.30)$$

Note that the normal form (2.28) as well as (2.30) are special cases of (2.27).

These systems will serve as a source of examples and they will be discussed at several places through the course of the paper.

2.1. The finite interval case, regularity. In this subsection we consider a finite interval $I = (a, b)$, $-\infty < a < b < \infty$. Moreover, we assume that $\mathcal{H}, B \in \mathcal{L}^1(a, b)$. In view of the previous discussion of gauge transformations w.l.o.g. we may assume that $J(x) = J(0) =: J$ is constant. We denote by $Y(\cdot, \lambda)$ the solution of (2.11) with $x_0 = a$.

We introduce the linear map

$$\delta_\lambda : \mathcal{L}^2_{\mathcal{H}}(I) \longrightarrow \mathbb{C}^n, \quad g \mapsto JY(b, \lambda)^{-1}(K_\lambda g)(b) = \int_a^b Y(t, \bar{\lambda})^* \mathcal{H}(t)g(t)dt. \quad (2.31)$$

Obviously, δ_λ induces a map on $L^2_{\mathcal{H}}(I)$. We will be sloppy here and do not distinguish between δ_λ and its induced map on $L^2_{\mathcal{H}}(I)$. For δ_0 we just write δ . Note that since δ_λ is continuous and since the target space \mathbb{C}^n is finite-dimensional we have

$$\text{im } \delta_\lambda = \delta_\lambda(\mathcal{L}^2_{\mathcal{H}, \text{comp}}(I)). \quad (2.32)$$

We have even more:

LEMMA 2.6. $\mathcal{L}^2_{\mathcal{H}, \text{comp}}(I) \cap \ker \delta_\lambda$ is dense in $\ker \delta_\lambda$.

PROOF. Let $g_1, \dots, g_k \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$ such that $\delta_\lambda(g_1), \dots, \delta_\lambda(g_k)$ is a basis of $\text{im } \delta_\lambda$. Then we have topological direct sum splittings

$$\begin{aligned}\mathcal{L}_{\mathcal{H}, \text{comp}}^2(I) &= (\mathcal{L}_{\mathcal{H}, \text{comp}}^2(I) \cap \ker \delta_\lambda) \oplus \langle g_1, \dots, g_k \rangle, \\ \mathcal{L}_{\mathcal{H}}^2(I) &= \ker \delta_\lambda \oplus \langle g_1, \dots, g_k \rangle.\end{aligned}\tag{2.33}$$

This implies the claim. \square

COROLLARY 2.7. *Let $\{f, g\} \in \mathcal{S}_{\max}$. Then, for $\{\tilde{f}, \tilde{g}\}$ to be in S it is sufficient that $f(a) = f(b) = 0$.*

PROOF. $f(a) = f(b) = 0$ implies $g \in \ker \delta$ and, in view of the previous lemma, we may choose a sequence $(g_n) \subset \ker \delta \cap \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$ with $g_n \rightarrow g$ in $\mathcal{L}_{\mathcal{H}}^2(I)$. Then $Kg_n \in \text{AC}_{\text{comp}}(I, \mathbb{C}^n)$ and $Kg_n \rightarrow Kg = f$ in $\mathcal{L}_{\mathcal{H}}^2(I)$. Thus $\{\widetilde{Kg_n}, \tilde{g}_n\} \in S_{\min}$ and $\{\widetilde{Kg_n}, \tilde{g}_n\} \rightarrow \{\tilde{f}, \tilde{g}\}$. \square

We put

$$\Phi(\lambda) = \Phi(\mathcal{S}, \lambda) = \int_a^b Y(x, \lambda)^* \mathcal{H}(x) Y(x, \lambda) dx.\tag{2.34}$$

For $\Phi(0)$ we just write Φ .

LEMMA 2.8 (cf. [18, Thm. 1.1]). *$\ker \Phi(\lambda), \text{im } \Phi(\lambda)$ are independent of λ , in particular $\text{rank } \Phi(\lambda)$ is independent of λ .*

PROOF. Fix $\lambda_0, \lambda \in \mathbb{C}$ and consider $\xi \in \ker \Phi(\lambda)$. Then we have

$$\int_a^b \xi^* Y(x, \lambda)^* \mathcal{H}(x) Y(x, \lambda) \xi dx = 0\tag{2.35}$$

and hence $\mathcal{H}(x) Y(x, \lambda) \xi = 0$ for almost all $x \in I$. Moreover, the function $f(x) = Y(x, \lambda) \xi$ satisfies the differential equation

$$Jf'(x) + Bf(x) = \lambda \mathcal{H}(x) f(x) = \lambda_0 \mathcal{H}(x) f(x)\tag{2.36}$$

for almost all $x \in \mathbb{R}$. Thus, by the uniqueness theorem for first order differential equations we have $f(x) = Y(x, \lambda_0) f(a) = Y(x, \lambda_0) \xi$. Moreover, since $\xi \in \ker \Phi(\lambda)$,

$$0 = \xi^* \Phi(\lambda) \xi = \int_a^b f(x)^* \mathcal{H}(x) f(x) dx = \xi^* \Phi(\lambda_0) \xi.\tag{2.37}$$

Since $\Phi(\lambda_0) \geq 0$ we infer $\xi \in \ker \Phi(\lambda_0)$.

Since λ_0, λ were arbitrary we have proved that $\ker \Phi(\lambda)$ is independent of λ . This implies the rest of the assertions. \square

The rank of Φ will play a crucial role, thus we put

$$\text{rank}(\mathcal{S}) := \text{rank}(S) := \text{rank}(\Phi).\tag{2.38}$$

LEMMA 2.9. $\text{im } \delta_\lambda = \text{im } \Phi = \{\xi \in \mathbb{C}^n \mid \mathcal{H}Y\xi = 0 \text{ a.e.}\}^\perp$.

Moreover, we have an orthogonal sum decomposition

$$\mathcal{L}_{\mathcal{H}}^2(I) = \ker \delta_\lambda \oplus \{Y(\cdot, \bar{\lambda})\xi \mid \xi \in \text{im } \Phi\}. \quad (2.39)$$

PROOF. First we prove (2.39). For any $\xi \in \mathbb{C}^n$ and $g \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$ one has

$$\begin{aligned} \langle \xi, \delta_\lambda(g) \rangle &= \int_a^b \xi^* Y(x, \bar{\lambda})^* \mathcal{H}(x) g(x) dx \\ &= \int_a^b (Y(x, \bar{\lambda})\xi)^* \mathcal{H}(x) g(x) dx = \langle Y(\cdot, \bar{\lambda})\xi, g \rangle_{\mathcal{H}}, \end{aligned} \quad (2.40)$$

hence $\delta_\lambda^*(\xi) = Y(\cdot, \bar{\lambda})\xi$. We note that $Y(\cdot, \bar{\lambda})\xi = 0$ in $L_{\mathcal{H}}^2(I)$ (that is $\mathcal{H}Y(\cdot, \bar{\lambda})\xi = 0$) for $\xi \in (\text{im } \delta_\lambda)^\perp$. Thus one infers

$$\mathcal{L}_{\mathcal{H}}^2(I) = \ker \delta_\lambda \oplus \text{im } \delta_\lambda^* = \ker \delta_\lambda \oplus \{Y(\cdot, \bar{\lambda})\xi \mid \xi \in \text{im } \delta_\lambda\}. \quad (2.41)$$

It follows that each $g \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$ admits a unique decomposition

$$g = g_0 + Y(\cdot, \bar{\lambda})\xi_g, \quad g_0 \in \ker \delta_\lambda, \quad \xi_g \in \text{im } \delta_\lambda, \quad (2.42)$$

where ξ_g is the unique element in $\text{im } \delta_\lambda$ such that $\delta_\lambda(Y(\cdot, \bar{\lambda})\xi_g) = \delta_\lambda(g)$. Furthermore,

$$\delta_\lambda(g) = \int_a^b Y(x, \bar{\lambda})^* \mathcal{H}(x) Y(x, \bar{\lambda}) \xi_g dx = \Phi(\bar{\lambda})\xi_g, \quad g \in \mathcal{L}_{\mathcal{H}}^2(I).$$

Hence $\text{im } \delta_\lambda \subset \text{im } \Phi(\bar{\lambda}) = \text{im } \Phi$. Since the opposite inclusion is obvious one gets $\text{im } \delta_\lambda = \text{im } \Phi$. In view of (2.41) this relation implies (2.39). To complete the proof it remains to note that $\ker \Phi = \{\xi \in \mathbb{C}^n \mid \mathcal{H}Y\xi = 0 \text{ a.e.}\}$. \square

PROPOSITION 2.10. (1) For all $\lambda \in \mathbb{C}$ we have

$$\text{im}(S_{\max} - \lambda) = L_{\mathcal{H}}^2(I),$$

$$\text{im}(S - \lambda) = \pi(\ker \delta_\lambda) = \left\{ \pi g \mid g \in \mathcal{L}_{\mathcal{H}}^2(I), \int_a^b Y(x, \bar{\lambda})^* \mathcal{H}(x) g(x) dx = 0 \right\},$$

$$\ker(S - \lambda) = \{0\},$$

$$\ker(S_{\max} - \lambda) = \{\pi Y(\cdot, \lambda)\xi \mid \xi \in \text{im } \Phi\} \simeq \text{im } \Phi.$$

(2) If $\{\tilde{f}, \tilde{g}\} \in S_{\max}$ then for each representative $g \in \tilde{g}$ there exists $f \in \tilde{f}$, $f \in \text{AC}(I, \mathbb{C}^n)$, such that $Jf' + Bf = \mathcal{H}g$. In particular $\pi_2(\mathcal{S}_{\max}) := (\pi \oplus \pi)(\mathcal{S}_{\max}) = S_{\max}$.

(3) $\pi_2(\{\{f, g\} \in \mathcal{S}_{\max} \mid f(a) = f(b) = 0\}) = S$. Moreover,

$$\mathcal{S} = (\pi_2^{-1}S) \cap \mathcal{S}^* = \{\{f, g\} \in \mathcal{S}^* \mid f(a) \in \ker \Phi, f(b) = Y(b)f(a)\}.$$

PROOF. (1) If $g \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$ is arbitrary then $\{\widetilde{K_\lambda g}, \tilde{g}\} \in (S_{\max} - \lambda)$ and we have proved that $\text{im}(S_{\max} - \lambda) = L_{\mathcal{H}}^2(I)$.

If $g \in \ker \delta_\lambda$ then by Corollary 2.7 we have $\{\widetilde{K_\lambda g}, \tilde{g}\} \in (S - \lambda)$, thus $\ker \delta_\lambda \subset \text{im}(S - \lambda)$. Since $\text{im}(S_{\min} - \lambda) \subset \ker \delta_\lambda$ by definition and since δ_λ is continuous we conclude that

$\text{im}(S - \lambda) \subset \overline{\text{im}(S_{\min} - \lambda)} \subset \ker \delta_\lambda$. We have proved $\ker \delta_\lambda = \text{im}(S - \lambda)$. Furthermore we infer $\ker(S - \lambda) = \text{im}(S_{\max} - \bar{\lambda})^\perp = \{0\}$ and $\ker(S_{\max} - \lambda) = \text{im}(S - \bar{\lambda})^\perp = (\ker \delta_{\bar{\lambda}})^\perp = \{\pi Y(\cdot, \lambda)\xi \mid \xi \in \text{im } \Phi\}$, in view of (2.39).

(2) Let $\{\tilde{f}, \tilde{g}\} \in S_{\max}$ and let $f \in \tilde{f}, g \in \tilde{g}$. We put $f_1(x) := Kg(x)$. Then $\{\tilde{f} - \tilde{f}_1, 0\} \in S_{\max}$, i.e. $\tilde{f} - \tilde{f}_1 \in \ker S_{\max}$. Consequently, there is a $\xi \in \text{im } \Phi$ such that $\tilde{f} = \tilde{f}_1 + Y\xi$ and hence $f_2 := f_1 + Y\xi$ is an absolute continuous representative of \tilde{f} which satisfies $Jf'_2 + Bf_2 = \mathcal{H}g$.

(3) Let $\{\tilde{f}, \tilde{g}\} \in S_{\max}$ with representatives $\{f, g\} \in \mathcal{S}_{\max}$. Then

$$f(x) = Y(x)f(a) + Kg(x). \quad (2.43)$$

If $\{\tilde{f}, \tilde{g}\} \in S$ then by (1) we have $g \in \ker \delta$ and hence $f(b) = Y(b)f(a)$. Moreover, $\{\tilde{Kg}, \tilde{g}\} \in S$ and thus $\{Yf(a), 0\} \in \ker S = \{0\}$. This implies $\mathcal{H}Yf(a) = 0$ a.e. and thus $f(a) \in \ker \Phi$.

Conversely, let $f(b) = Y(b)f(a)$ and $f(a) \in \ker \Phi$. Then $Kg = f - Yf(a)$ represents the same element $\tilde{f} \in L^2_{\mathcal{H}}(I)$ as f . Moreover $f(b) = Y(b)f(a)$ implies $\delta(g) = 0$, hence $\{\tilde{f}, \tilde{g}\} = \{\tilde{Kg}, \tilde{g}\}$. Since $Kg(a) = Kg(b) = 0$ this argument also shows $\pi_2(\{\{f, g\} \in \mathcal{S}_{\max} \mid f(a) = f(b) = 0\}) = S$. \square

2.2. Arbitrary intervals. Now we consider an arbitrary, finite or infinite, interval $I \subset \mathbb{R}$. Let J, B, \mathcal{H} be as in (2.2) with $J = J(0)$ constant. We fix a point $x_0 \in I$ and denote by $Y(x, \lambda)$ the solution (2.11). For any finite subinterval $\tilde{I} \subset I^\circ$, $I^\circ := I \setminus \partial I$, we consider the matrix

$$\Phi_{\tilde{I}}(\lambda) := \int_{\tilde{I}} Y(x, \lambda)^* \mathcal{H}(x) Y(x, \lambda) dx. \quad (2.44)$$

In view of Lemma 2.8 the range of $\Phi_{\tilde{I}}(\lambda)$ is independent of λ and as before we write $\Phi_{\tilde{I}}$ instead of $\Phi_{\tilde{I}}(0)$. Note, however, that $\Phi_{\tilde{I}}(\lambda)$ depends on the choice of the base point x_0 . $\tilde{I} \mapsto \Phi_{\tilde{I}}$ is an increasing map with values in the positive semi-definite matrices. Moreover, in view of (2.44) $\Phi_{\tilde{I}}$ depends continuously on the endpoints of \tilde{I} . Since the rank is a lower semi-continuous function on the space of $n \times n$ matrices we infer that there exists a compact interval $I_0 \subset I^\circ$ such that for any compact interval $I_0 \subset \tilde{I} \subset I^\circ$ we have

$$\text{im } \Phi_{I_0} = \text{im } \Phi_{\tilde{I}}. \quad (2.45)$$

We then put (cp. (2.38))

$$\text{rank}(S) := \text{rank}(\mathcal{S}) := \text{rank } \Phi_{I_0}. \quad (2.46)$$

Somewhat sloppy, in view of (2.45), we will write $\ker \Phi, \text{im } \Phi$ for $\ker \Phi_{I_0}, \text{im } \Phi_{I_0}$. For $g \in \mathcal{L}^2_{\mathcal{H}, \text{comp}}(I^\circ)$ we put

$$\delta_\lambda(g) := \int_I Y(x, \bar{\lambda})^* \mathcal{H}(x) g(x) dx. \quad (2.47)$$

LEMMA 2.11. *Let $k = \text{rank}(S)$. Then there exist $g_1, \dots, g_k \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$ such that there is a direct sum decomposition*

$$\mathcal{L}_{\mathcal{H}, \text{comp}}^2(I) = \ker \delta_\lambda \oplus \langle g_1, \dots, g_k \rangle. \quad (2.48)$$

PROOF. In view of Lemma 2.9 and the previous considerations we have $\text{im } \delta_\lambda = \text{im } \Phi_{I_0}$. Hence, from (2.33) we infer that we may choose $g_1, \dots, g_k \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I_0)$ such that $\delta_\lambda(g_1), \dots, \delta_\lambda(g_k)$ is a basis of $\text{im } \delta_\lambda$. This implies the assertion. \square

Now we are in the position to prove the analogue of Proposition 2.10 for general intervals.

PROPOSITION 2.12. *Let \mathcal{S} be the symmetric linear relation induced by the first order system (2.1) on an arbitrary interval I . Then:*

- (1) $\text{im}(\mathcal{S}_{\min} - \lambda) \supset \ker \delta_\lambda$. Moreover, if $I = [0, b)$ is left-closed (resp. $I = (a, 0]$ right-closed) then $\text{im}(S_{\max} - \lambda)$ is dense in $L_{\mathcal{H}}^2(I)$ and $\ker(S - \lambda) = \{0\}$.
- (2) If $\{\tilde{f}, \tilde{g}\} \in S_{\max}$ then for each representative $g \in \tilde{g}$ there exists $f \in \tilde{f}$, $f \in \text{AC}(I, \mathbb{C}^n)$, such that $Jf' + Bf = \mathcal{H}g$. In particular $\pi_2(\mathcal{S}_{\max}) = S_{\max}$.
- (3) Let $I = \mathbb{R}_+$ and let Φ_0 and Φ_1 be the matrices constructed in (2.44)–(2.46) with respect to the base point $c \in [0, \infty]$ and the intervals $[0, c]$ and $[c, \infty)$ respectively. Suppose also that $\text{im}(\Phi_0) = \text{im}(\Phi_1)$. Then for each $\xi \in \text{im}(J^{-1}\Phi)$ there exists $\{f, g\} \in \mathcal{S}_{\max}$ with compact support such that $f(c) = \xi$. Moreover, $\{f, g\} \in \mathcal{S}_{\min}$ if $c > 0$.
- (4) Let $I = \mathbb{R}_\pm$ and let $\{\tilde{f}, \tilde{g}\} \in S$ with representatives $\{f, g\} \in \mathcal{S}_{\max}$. Then $f(0) \in \ker \Phi$. Moreover, $\pi_2(\{\{f, g\} \in \mathcal{S} \mid f(0) = 0\}) = S$.

PROOF. For simplicity we will give the proof for $\lambda = 0$.

(1) Let $I = [0, b)$ be left-closed and let $g \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$. Then choose $c > \max(\text{supp } g)$ and put

$$f(x) := Y(x) \int_a^x J^{-1}Y(t)^* \mathcal{H}(t)g(t)dt. \quad (2.49)$$

Since I is left-closed we then have $\{\tilde{f}, \tilde{g}\} \in S_{\max}$ and hence $L_{\mathcal{H}, \text{comp}}^2(I) \subset \text{im } S_{\max}$. Thus S_{\max} has dense range and consequently $\ker S = \{0\}$.

The same construction shows for any interval I that if $g \in \ker \delta \cap \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I^\circ)$ then the function f has compact support in I° and thus $\text{im } \mathcal{S}_{\min} \supset \ker \delta$.

(2) Let $f_0 \in \tilde{f}$ be any representative and put $f_1(x) = Y(x) \int_{x_0}^x J^{-1}Y(y)^* \mathcal{H}(y)g(y)dy$. Then f_1 is absolutely continuous. Using integration by parts and (2.13) one obtains for any pair $\{\varphi, \psi\} \in \mathcal{S}_{\min}$

$$\int_I f_1^* \mathcal{H} \psi = \int_I g^* \mathcal{H} \varphi = \int_I f_0^* \mathcal{H} \psi. \quad (2.50)$$

By (1) we have $\text{im } \mathcal{S}_{\min} \supset \ker \delta$, thus (2.50) implies

$$\int_I (f_0 - f_1)^* \mathcal{H} \psi = 0, \quad \text{for all } \psi \in \ker \delta. \quad (2.51)$$

Since the g_j in Lemma 2.11 satisfy $\text{supp}(g_j) \subset I_0$ we apply Lemma 2.9 and Lemma 2.11 to conclude that there is a $\xi \in \text{im } \Phi$ such that for all $\psi \in \mathcal{L}_{\mathcal{H}, \text{comp}}^2(I)$ one has

$$\int_I (f_0 - f_1 - Y\xi)^* \mathcal{H}\psi = 0. \quad (2.52)$$

Note that by integration by parts one has $\int_I \xi^* Y(x)^* \mathcal{H}(x) u(x) dx = 0$ for all $u \in \ker \delta$, even if $\text{supp}(u) \cap (I \setminus I_0) \neq \emptyset$. (2.52) implies that $f = f_1 + Y\xi$ is an absolute continuous representative of \tilde{f} with $Jf' + Bf = \mathcal{H}g$.

(3) We may assume that $I_0 = [a_0, c]$ and $I_1 = [c, a_1]$ where $a_0 > 0$. Then choose $\eta_0, \eta_1 \in \mathbb{C}^n$ satisfying $\xi = J^{-1}\Phi_0\eta_0 = -J^{-1}\Phi_1\eta_1$ and put

$$g(t) = \begin{cases} \chi_0(t)Y(t)\eta_0, & t \in [0, c), \\ \chi_1(t)Y(t)\eta_1, & t \in [c, \infty). \end{cases}$$

Here χ_0 and χ_1 are the characteristic functions of the intervals $I_0 = [a_0, c)$ and $I_1 = [c, a_1]$ respectively. Then we define f by (2.49) with a replaced by a_1 . It is clear that $\text{supp } f \subset [0, a_1]$ and

$$f(c) = J^{-1} \int_{a_1}^c Y(t)^* \mathcal{H}(t) g(t) dt = -J^{-1}\Phi_1\eta_1 = \xi.$$

Furthermore, for $x \in [0, a_0]$ one gets

$$\begin{aligned} f(x) &= Y(x)J^{-1} \int_{a_1}^x Y^*(t)\mathcal{H}(t)g(t)dt \\ &= -Y(x)J^{-1} \left[\int_{a_0}^c Y^*(t)\mathcal{H}(t)Y(t)dt \eta_0 + \int_c^{a_1} Y^*(t)\mathcal{H}(t)Y(t)dt \eta_1 \right] \\ &= -Y(x)J^{-1}[\Phi_0\eta_0 + \Phi_1\eta_1] = 0. \end{aligned}$$

(4) Let $\xi \in \text{im } J^{-1}\Phi$. According to (3) we may choose $\{\varphi, \psi\} \in \mathcal{S}_{\max}$ with compact support such that $\varphi(0) = \xi$. For each $\{\tilde{f}, \tilde{g}\} \in S$ we have on the one hand $(\varphi, g)_{\mathcal{H}} = (\psi, f)_{\mathcal{H}}$. Since φ, ψ have compact support we may integrate by parts and thus find

$$0 = \varphi(0)^* Jf(0) = -\langle J\varphi(0), f(0) \rangle = \langle J\xi, f(0) \rangle.$$

Thus $f(0)$ is orthogonal to $\text{im } \Phi$, that is $f(0) \in \ker \Phi$.

To prove the last assertion let $\{\tilde{f}, \tilde{g}\} \in S$ with representatives $\{f, g\} \in \mathcal{S}$. Then $f(0) \in \ker \Phi$ and hence $f_1 := f - Yf(0)$ is an absolute continuous representative of f . Moreover, $Jf'_1 + Bf_1 = g$ and $f_1(0) = 0$. Consequently, $\{\tilde{f}, \tilde{g}\} = \{\tilde{f}_1, g\}$ and $\{f_1, g\} \in S$. \square

REMARK 2.13. The converse of (4) does not hold without further assumptions. Roughly speaking the system has to be "in the limit point case" at infinity. We will give criteria under which this is true.

2.3. Definite first order systems.

DEFINITION 2.14. The system (2.1) is said to be *definite* on I if $\ker \Phi = \{0\}$. In other words there is a compact subinterval $I_0 \subset I^\circ$ such that for all intervals $I_0 \subset \tilde{I} \subset I^\circ$ and all $\lambda \in \mathbb{C}$ the matrix $\Phi_{\tilde{I}}(\lambda)$ (cf. (2.44), Lemma 2.8) is invertible.

In other words, the system (2.1) is definite if 0 is the only solution of

$$Jf' + Bf = 0, \quad \mathcal{H}f = 0$$

in $\mathcal{L}_{\mathcal{H}}^2(I)$.

The property of a system (2.1) to be definite is gauge invariant. For a canonical system $(J = J(0), B = 0)$ this property may be reformulated solely in terms of the Hamiltonian \mathcal{H} . Namely, it is shown in [11] and [18] that a canonical system is definite iff the Hamiltonian \mathcal{H} is of positive type, that is $\int_{I_0} \mathcal{H}$ is invertible for some I_0 .

Note also that the system (2.1) is definite for arbitrary B and J if the Hamiltonian \mathcal{H} is positive definite on a subset of positive Lebesgue measure. We emphasize however that for a general system (2.1) being definite is a property of the system and depends on J, B , too. The two examples show that the invertibility of $\int_{I_0} \mathcal{H}$ is unrelated to \mathcal{H} being of positive type.

The usefulness of the notion of definiteness mainly stems from the following fact:

PROPOSITION 2.15. *Assume that the system (2.1) is definite. Let $\{\tilde{f}, \tilde{g}\} \in S_{\max}$. Moreover let $\{f_j, g_j\} \in \mathcal{S}_{\max}, j = 1, 2$, be representatives of $\{\tilde{f}, \tilde{g}\}$, i.e. $f_j \in \tilde{f}, g_j \in \tilde{g}$. Then $f_1 = f_2$.*

REMARK 2.16. Note that the Proposition does not say that \tilde{f} has exactly one absolute continuous representative. In fact it is easy to see that this is false. See the third example below.

However, Proposition 2.15 allows to speak of the value of \tilde{f} at a point. I.e. for $x \in \mathbb{R}$ put $\tilde{f}(x) := f(x)$, where $\{f, g\} \in \mathcal{S}_{\max}$ is a representative of $\{\tilde{f}, \tilde{g}\}$. Proposition 2.15 says that $\tilde{f}(x)$ is well-defined independently of the choice of $\{f, g\}$.

PROOF. Consider $\{\varphi, \psi\} := \{f_1 - f_2, g_1 - g_2\} \in \mathcal{S}_{\max}$. Then $\{\tilde{\varphi}, \tilde{\psi}\} = 0$ and hence

$$J\varphi' + B\varphi = 0, \quad \mathcal{H}\varphi = 0. \quad (2.53)$$

Then the definiteness implies $\varphi = 0$ and we are done. \square

EXAMPLE 2.17. 1. Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = -I_2, \quad \mathcal{H}(x) = \begin{pmatrix} \cos^2(x) & \sin(x)\cos(x) \\ \sin(x)\cos(x) & \sin^2(x) \end{pmatrix}, \quad (2.54)$$

and $I = [0, \pi]$. Then $\int_0^\pi \mathcal{H}(x)dx = \frac{\pi}{2}I_2$ is invertible. However, the function

$$f(x) = \begin{pmatrix} \sin(x) \\ -\cos(x) \end{pmatrix} \quad (2.55)$$

satisfies $Jf' + Bf = 0$ and $\mathcal{H}f = 0$. Thus the system is not definite.

Note, that for this system we have

$$Y(x) = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}. \quad (2.56)$$

Using this as gauge (cf. (2.15)) we obtain the corresponding canonical system $S(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$ with $\tilde{J} = J, \tilde{B} = 0, \tilde{\mathcal{H}} = \text{diag}(1, 0)$. It is clear that this system is not definite.

2. Let $V \in L^1(I)$ and put

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} V & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{H}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.57)$$

Then it is easy to check that $L^2_{\mathcal{H}}(I) \simeq L^2(I)$, and the equation $Jf' + Bf = \mathcal{H}g$ is equivalent to $-f_1'' + Vf_1 = g_1$. This shows that the system is equivalent to the Schrödinger operator $-\frac{d^2}{dx^2} + V$ on the interval I .

Now assume that I is a finite interval. Then $\int_I \mathcal{H}$ is of rank one and hence not invertible. We claim, however, that the system is definite. Namely, let $Jf' + Bf = 0$ and $\int_I f^* \mathcal{H} f = 0$. Then $f_1 = 0$ and since $f_2 = f_1'$ we also have $f_2 = 0$.

Another way of seeing this is to look at the fundamental system Y . Y is a Wronski matrix

$$Y = \begin{pmatrix} f & g \\ f' & g' \end{pmatrix}, \quad (2.58)$$

thus

$$\tilde{\mathcal{H}} = Y^* \mathcal{H} Y = \begin{pmatrix} f^2 & fg \\ fg & g^2 \end{pmatrix}. \quad (2.59)$$

Since f, g are linearly independent the Cauchy–Schwarz–Bunyakovskii inequality yields $\int_I \tilde{\mathcal{H}} > 0$. This example is a special case of Example 2.5. See also Proposition 2.18 for a more general result on definiteness.

3. In 2. consider the special case $V = 0$. Put $f := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, g = 0$. Then $\{f, g\} \in \mathcal{S}_{\max}$. However, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a second absolute continuous representative of \tilde{f} . This is an example for the claim made in Remark 2.16.

The last example is a special case of the following definiteness result for systems of the form (2.28), (2.30).

PROPOSITION 2.18. *Let $I \subset \mathbb{R}$ be an interval. We consider the system $S(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$, where*

$$\tilde{J} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} V & B \\ B^* & -A \end{pmatrix}, \quad \tilde{\mathcal{H}} = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & 0 \end{pmatrix} \quad (2.60)$$

are as in (2.30). Assume that the set $I_0 := \{x \in I \mid \det(A(x)\mathcal{H}(x)) \neq 0\}$ has positive Lebesgue measure. Then the system $S(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$ is definite.

PROOF. Consider $f \in \mathcal{L}_{\mathcal{H}}^2(I) \cap \text{AC}(I, \mathbb{C}^{2n})$ satisfying

$$\tilde{J}f' + \tilde{B}f = 0, \quad \int_I f^* \tilde{\mathcal{H}} f = 0. \quad (2.61)$$

We have to show that $f = 0$. (2.61) translates into

$$f_1' + B^* f_1 - A f_2 = 0, \quad (2.62)$$

$$-f_2' + B f_2 + V f_1 = 0, \quad (2.63)$$

$$\int_I f_1^* \mathcal{H} f_1 = 0. \quad (2.64)$$

(2.64) implies that $\mathcal{H} f_1 = 0$ a.e. Thus the set $I_1 := \{x \in I \mid \det(A(x)\mathcal{H}(x)) \neq 0, \mathcal{H}(x)f_1(x) = 0\} = \{x \in I \mid \det(A(x)\mathcal{H}(x)) \neq 0, f_1(x) = 0\}$ has positive Lebesgue measure. A set of positive Lebesgue measure contains an accumulation point of itself; the reason is that a subset of the reals which does not contain an accumulation point of itself is at most countable. So let $x_0 \in I_1$ be an accumulation point of I_1 . Then $f_1(x_0) = f_1'(x_0) = 0$ and by (2.62) $A(x_0)f_2(x_0) = 0$. Since $A(x_0)$ is invertible we infer $f_1(x_0) = f_2(x_0) = 0$ and hence $f(x_0) = 0$. Since f is a solution of the homogeneous first order equation $\tilde{f}' + \tilde{B}f = 0$ this implies $f = 0$. \square

2.4. Formal defect subspaces. In this section we present some results on the square-integrable solutions of the system

$$J(x)y'(x) + B(x)y(x) = \lambda \mathcal{H}(x)y(x). \quad (2.65)$$

Let

$$\begin{aligned} \mathcal{E}_\lambda(S) &:= \{f \in \mathcal{L}_{\mathcal{H}}^2(I) \cap \text{AC}(I, \mathbb{C}^n) \mid Jf' + Bf = \lambda \mathcal{H}f\} \\ &= \{f \in \mathcal{L}_{\mathcal{H}}^2(I) \mid \{f, \lambda f\} \in \mathcal{S}^*\} \\ &= \ker(S^* - \lambda), \end{aligned} \quad (2.66)$$

and denote by

$$\mathcal{N}_\pm(S) := \dim \mathcal{E}_{\pm i}(S) \quad (2.67)$$

the *formal* deficiency indices of the system (2.1). Furthermore, for a symmetric linear relation A in the Hilbert space \mathfrak{H} we denote by

$$\begin{aligned} E_\lambda(A) &:= \{f \in \mathfrak{H} \mid \{f, \lambda f\} \in A^*\} \\ &= \ker(A^* - \lambda), \end{aligned} \quad \lambda \in \mathbb{C}, \quad (2.68)$$

the defect subspace and by

$$N_\pm(A) := \dim E_{\pm i}(A) \quad (2.69)$$

the deficiency indices of A . It is well-known (see [1], [26]) that

$$\dim E_{\pm \lambda}(A) = N_\pm(A), \quad \lambda \in \mathbb{C}_+ := \{z \in \mathbb{C} \mid \text{Im } z > 0\}. \quad (2.70)$$

We present however two simple proofs of (2.70).

The first proof follows from the observation that the relation $A^* - \lambda$ is semi-Fredholm for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Thus $\dim E_\lambda(A)$ is locally constant on $\mathbb{C} \setminus \mathbb{R}$ (see [17]) and therefore $\dim E_{\pm\lambda}(A) = \dim E_{\pm i}(A)$ for $\lambda \in \mathbb{C}_+$. For another proof see Corollary 2.23 below.

There are situations in which it is clear that the formal defect spaces $\mathcal{E}_\lambda(S)$ and the defect spaces $E_\lambda(S)$ are isomorphic. This is, for instance, the case if $\mathcal{H}(x)$ is invertible for almost all $x \in I$. In general, the analogue of (2.70) for the dimensions of the formal defect subspaces $\mathcal{E}_\lambda(S)$ holds. However, this is less trivial. The only proof we know of so far is due to Kogan and Rofo-Beketov [18, Sec. 2]. It uses methods from complex analysis and is rather technical. Here we can give a very simple proof of this fact which is based on the regularity Theorem 2.4. Namely, the regularity Theorem allows to show a simple relation between the deficiency indices and the formal deficiency indices:

PROPOSITION 2.19. *Let S be a general symmetric system (2.1) on an interval $I \subset \mathbb{R}$. Then for $\lambda \in \mathbb{C}$ we have*

$$\dim \mathcal{E}_\lambda(S) = \dim E_\lambda(S) + n - \text{rank } S. \quad (2.71)$$

In particular, if the system is definite then $\dim \mathcal{E}_\lambda(S) = \dim E_\lambda(S)$.

PROOF. Consider $\tilde{f} \in E_\lambda(S)$. This means $\{\tilde{f}, \lambda \tilde{f}\} \in S_{\max}$ and in view of Theorem 2.4 there exists $f \in \tilde{f}, f \in \text{AC}(I, \mathbb{C}^n) \cap \mathcal{L}_{\mathcal{H}}^2(I)$ such that $Jf' + Bf = \lambda \mathcal{H}f$. Thus $f \in \mathcal{E}_\lambda(S)$. This shows that the quotient map $\pi : \mathcal{E}_\lambda(S) \rightarrow E_\lambda(S)$ is surjective.

Next let $\{f, \lambda f\} \in \ker \pi$. This means that $Jf' + Bf = \lambda \mathcal{H}f$ and $\tilde{f} = 0$. Thus $\mathcal{H}f = 0$. Hence $\ker \pi$ consists of the solutions of $Jf' + Bf = 0, \mathcal{H}f = 0$. This space is isomorphic to $\ker \Phi$ (cf. Subsections 2.1, 2.2) and hence $\dim \ker \pi = \dim \ker \Phi = n - \text{rank } S$ and we reach the conclusion. \square

The following result was proved by Kogan and Rofo-Beketov for the half-line [18, Theorem 2.1] and for systems on the line which are definite on both half-lines \mathbb{R}_\pm [18, Corollary 2.2]. For general non-definite systems it seems to be new.

PROPOSITION 2.20. *Let S be a general symmetric system (2.1) on an interval $I \subset \mathbb{R}$. Then*

$$\dim \mathcal{E}_{\pm\lambda}(S) = \dim \mathcal{E}_{\pm i}(S) =: \mathcal{N}_\pm(S), \quad \text{for } \lambda \in \mathbb{C}_+.$$

PROOF. This follows immediately from (2.70) and Proposition 2.19. \square

For completeness we note the case of a finite interval:

PROPOSITION 2.21. *Let $I = [a, b]$ be a finite interval and S the symmetric linear relation obtained from the first order system (2.1), where $B, \mathcal{H} \in \mathcal{L}^1(a, b)$. Then $\mathcal{N}_\pm(\mathcal{S}) = n$ and $N_\pm(S) = \text{rank}(S)$.*

In particular $\mathcal{N}_\pm(\mathcal{S}) = N_\pm(S) = n$ if the system S is definite.

PROOF. It is clear that the differential equation (2.1) has n linear independent solutions. Hence $\mathcal{N}_\pm(\mathcal{S}) = n$. From Proposition 2.10 we infer that $\widetilde{Y\xi}$ is nonzero if and only if $\xi \in \text{im } \Phi$. This implies $N_\pm(S) = \text{rank } \Phi$. \square

2.5. Definite systems and von Neumann formula. We start with the following generalization of the von Neumann formula.

PROPOSITION 2.22. *Let A be a closed symmetric linear relation in the Hilbert space \mathfrak{H} . Then for each pair $\{\lambda_1, \lambda_2\} \in \mathbb{C}_+ \times \mathbb{C}_-$ we have the following direct sum decomposition*

$$A^* = A \dot{+} \hat{E}_{\lambda_1} \dot{+} \hat{E}_{\lambda_2}, \quad \hat{E}_\lambda = \{ \{f, \lambda f\} \mid f \in E_\lambda \}. \quad (2.72)$$

$\dot{+}$ denotes a (non-orthogonal) direct sum of vector spaces.

PROOF. We put $A_{\lambda_1} := A \dot{+} \hat{E}_{\lambda_1}$. It is clear that A_{λ_1} is a closed dissipative extension of A , that is $A \subset A_{\lambda_1} \subset A^*$ and $\text{Im}(f, g) \geq 0$ for any $\{f, g\} \in A_{\lambda_1}$. In fact, we show that A_{λ_1} is a maximal dissipative relation in \mathfrak{H} . To prove this fact it suffices to check that $\bar{\lambda}_1 \in \rho(A_{\lambda_1})$, where $\rho(A_{\lambda_1})$ denotes the resolvent set of A_{λ_1} .

For a dissipative linear relation T and $\mu = \alpha - i\beta \in \mathbb{C}_-$ one has for $\{f, g\} \in T$

$$\|g - \mu f\|^2 = \|g - \alpha f\|^2 + 2\beta \text{Im}(f, g) + \beta^2 \|f\|^2 \geq \beta^2 \|f\|^2.$$

Hence $A_{\lambda_1} - \bar{\lambda}_1 I$ is injective with closed range and thus it suffices to verify that $\text{im}(A_{\lambda_1} - \bar{\lambda}_1 I)$ is dense in \mathfrak{H} .

Let φ be orthogonal to $\text{im}(A_{\lambda_1} - \bar{\lambda}_1 I)$, that is

$$\langle g - \bar{\lambda}_1 f, \varphi \rangle = 0 \quad \text{for } \{f, g\} \in A_{\lambda_1}. \quad (2.73)$$

In particular, we have for $\{f, g\} \in A$

$$\langle g, \varphi \rangle = \langle \bar{\lambda}_1 f, \varphi \rangle = \langle f, \lambda_1 \varphi \rangle.$$

Hence $\varphi \in E_{\lambda_1}$ and $\{\varphi, \lambda_1 \varphi\} \in \hat{E}_{\lambda_1}$. From the latter and (2.73) we infer $0 = \langle \lambda_1 \varphi - \bar{\lambda}_1 \varphi, \varphi \rangle = -2i \text{Im } \lambda_1 \|\varphi\|^2$. Hence $\varphi = 0$. Summing up, we have proved that $\mathbb{C}_- \subset \rho(A_{\lambda_1})$ and hence A_{λ_1} is maximal dissipative.

On the other hand for each proper extension $\tilde{A}, A \subset \tilde{A} \subset A^*$ the inclusion $\mu \in \rho(\tilde{A})$ is equivalent to the fact that \tilde{A} is transversal¹ to $A_\mu := A \dot{+} \hat{E}_\mu$ (see [25]). Hence A_{λ_1} and A_{λ_2} are transversal and this is equivalent to the direct sum decomposition (2.72). \square

Now we can give the second proof of (2.70).

COROLLARY 2.23 ([1], [26]). *With the previous notations we have for all $\lambda \in \mathbb{C}_+$*

$$\dim E_{\pm\lambda} = \dim E_{\pm i}. \quad (2.74)$$

PROOF. Let $\lambda_2 = -i$. It follows from (2.72), that for each $\lambda_1 \in \mathbb{C}_+$

$$\dim E_{\lambda_1} = \dim A^* / (A \dot{+} \hat{E}_{-i}).$$

\square

¹Two proper extensions A_1 and A_2 of A are called transversal if $A_1 \cap A_2 = A$ and $A_1 + A_2 = A^*$.

REMARK 2.24. 1. Formula (2.72) with $\lambda_2 = \bar{\lambda}_1$ is well-known [27], [10], [3]. For $\lambda_1 = i = \bar{\lambda}_2$ the direct sum (2.72) is orthogonal

$$A^* = A \oplus \hat{E}_i \oplus \hat{E}_{-i}. \quad (2.75)$$

2. The maximal dissipativity of the linear relation A_λ with $\lambda \in \mathbb{C}_+$ is well-known. We presented the proof for the sake of completeness. Note, however, that our proof of this fact as well as the proof of the well-known Corollary 2.23 is simpler and shorter than the known ones.

We continue by noting a simple lemma which is a generalization of a well-known result (cf. [26]) on symmetric operators to the case of symmetric linear relations.

LEMMA 2.25. *Let A be a closed symmetric linear relation in the Hilbert space \mathfrak{H} and $\ker(A - aI) = \{0\}$ for some $a \in \mathbb{R}$. Then*

$$\dim E_a(A) = \dim \ker(A^* - aI) \leq N_\pm(A). \quad (2.76)$$

PROOF. Similar to the proof of Proposition 2.22 we put

$$\tilde{A}_a := A \dot{+} \hat{E}_a(A), \quad \hat{E}_a(A) := \{\{f, af\} \mid f \in E_a(A)\}.$$

It is clear that \tilde{A}_a is a symmetric extension of A and the subspaces A and $\hat{E}_a(A)$ are linearly independent since $\ker(A - aI) = \{0\}$. Therefore $\dim(\tilde{A}_a/A) = \dim E_a(A)$. On the other hand the von Neumann formula for linear relations (2.75) yields $\dim(\tilde{A}_a/A) \leq \min(N_+, N_-)$. Combining these relations we obtain (2.76). \square

We return to the discussion of the relation $S = S(J, B, \mathcal{H})$. Denote by $\text{Exts}(\mathcal{S})$ and $\text{Exts}(S)$ the set of closed symmetric extensions of \mathcal{S}_{\min} and S_{\min} respectively:

$$\begin{aligned} \text{Exts}(\mathcal{S}) &= \{\tilde{\mathcal{S}} \mid \mathcal{S}_{\min} \subset \tilde{\mathcal{S}} \subset \mathcal{S}_{\max}, \tilde{\mathcal{S}} \text{ is closed and symmetric}\}, \\ \text{Exts}(S) &= \{\tilde{S} \mid S_{\min} \subset \tilde{S} \subset S_{\max}, \tilde{S} \text{ is closed and symmetric}\}. \end{aligned} \quad (2.77)$$

PROPOSITION 2.26. *Assume that the system (2.1) is definite on \mathbb{R}_+ . Then:*

- (1) *The quotient map π maps $E_\lambda(S)$ isomorphically onto $\mathcal{E}_\lambda(S)$ for each $\lambda \in \mathbb{C}$ and consequently $\mathcal{N}_\pm(S) = N_\pm(S)$.*
- (2) *For each $\xi \in \mathbb{C}^n$ and each $a \in [0, \infty]$ there exists $\{f, g\} \in \mathcal{S}^*$ with compact support such that $f(a) = \xi$. If $a > 0$ and the system S is definite both on $[0, a]$ and $[a, \infty]$, then $\{f, g\}$ can be chosen such that $\{f, g\} \in \mathcal{S}_{\min}$.*
- (3) *If $\{\tilde{f}, \tilde{g}\} \in S$ with representatives $\{f, g\} \in \mathcal{S}_{\max}$ then $f(0) = 0$, that is*

$$\mathcal{D}(\mathcal{S}) \subset \{f \in \text{AC}(\mathbb{R}_+, \mathbb{C}^n) \mid f(0) = 0\}.$$

- (4) *The quotient map $\pi_2 := \pi \oplus \pi$ maps \mathcal{S}^* and \mathcal{S} isomorphically onto S_{\max} and S respectively.*
- (5) *For each pair $\{\lambda_1, \lambda_2\} \in \mathbb{C}_+ \times \mathbb{C}_-$ the following analogue of the von Neumann formula holds true*

$$\mathcal{S}_{\max} = \mathcal{S} \dot{+} \hat{\mathcal{E}}_{\lambda_1}(\mathcal{S}) \dot{+} \hat{\mathcal{E}}_{\lambda_2}(\mathcal{S}), \quad \hat{\mathcal{E}}_\lambda(\mathcal{S}) := \{\{f, \lambda f\} \mid f \in \mathcal{E}_\lambda\}. \quad (2.78)$$

For $\lambda_1 = i = \bar{\lambda}_2$ the direct sum decomposition (2.78) is orthogonal.

- (6) The quotient map $\pi_2 := \pi \oplus \pi$ induces a bijective correspondence between the sets $\text{Exts}(\mathcal{S})$ and $\text{Exts}(\widetilde{\mathcal{S}})$. Moreover, \mathcal{S} is self-adjoint iff $\widetilde{\mathcal{S}}$ is self-adjoint.
 (7) For each $a \in \mathbb{R}$ the following inequality holds

$$\mathcal{N}_\pm(\mathcal{S}) \geq \dim \mathcal{E}_a(\mathcal{S}) = N_a(S). \quad (2.79)$$

PROOF. (1) has been established in the proof of Proposition 2.19.

(2) is implied by Proposition 2.12 (3) since $\text{im } \Phi = \mathbb{C}^n$.

(3) is a special case of Proposition 2.12 (4) since $\ker \Phi = 0$.

(4) Injectivity of the map $\pi_2 : \mathcal{S}^* \rightarrow S_{\max}$ follows again from the assumption that S is definite. Indeed, let $\{\tilde{f}, \tilde{g}\} \in S_{\max}$, $f_k \in \tilde{f}$, $\{f_k, g\} \in \mathcal{S}^*$, $k = 1, 2$. Then $f := f_1 - f_2$ satisfies the homogeneous equation $Jf' + Bf = 0$, that is $f \in E_0(\mathcal{S})$. Since $f_1, f_2 \in \tilde{f}$ we have $\mathcal{H}f = 0$ and therefore $\int_I f^*(x)\mathcal{H}(x)f(x)dx = 0$. Since S is definite the latter implies $f = 0$.

Surjectivity has been established in Proposition 2.12.

(5) is a consequence of (1),(4) and Proposition 2.22.

(6) W.l.o.g. we may assume $N_+ \leq N_-$. By definiteness we then have $\mathcal{N}_+ = N_+ \leq N_- = \mathcal{N}_-$. It follows from the von Neumann formula (2.78) with $\lambda_1 = \bar{\lambda}_2 = i$ that each symmetric extension $\widetilde{\mathcal{S}} \supset \mathcal{S}$ is given by the second Neumann formula

$$\widetilde{\mathcal{S}} = \mathcal{S} \dot{+} \{(I + V)\mathcal{E}'_i, i(I - V)\mathcal{E}'_i\}, \quad (2.80)$$

where $\mathcal{E}'_i \subset \mathcal{E}_i$ is a linear subspace and V is an isometric operator from \mathcal{E}'_i onto $V\mathcal{E}'_i \subset \mathcal{E}_{-i}$. The corresponding symmetric extension $\widetilde{S} \supset S$ is given by

$$\widetilde{S} = S \dot{+} \{(I + V)E'_i, i(I - V)E'_i\}, \quad E'_i := \pi \mathcal{E}'_i \subset E_i. \quad (2.81)$$

It is clear from (1) that this establishes the asserted bijective correspondence. Cf. also Proposition 2.15

(7) We know from Proposition 2.12 (1) that $\ker(S - aI) = \{0\}$. Lemma 2.25 and (1) now imply

$$\mathcal{N}_\pm(\mathcal{S}) = N_\pm(S) \geq \dim \ker(S_{\max} - aI) = \dim \mathcal{E}_a(\mathcal{S}). \quad (2.82)$$

□

3. Essential self-adjointness on the line. First approach.

3.1. Preliminaries and a first criterion for essential self-adjointness. In this section we study the system (2.1) on the real line and discuss essential self-adjointness. For the moment let $I \subset \mathbb{R}$ be an interval and $S = S(J, B, \mathcal{H})$ be the symmetric linear relation of the first order system (2.1). Let

$$\lambda_j(x) := \max_{V \subset \mathbb{C}^n, \dim V = j-1} \min \{ \langle \mathcal{H}(x)\xi, \xi \rangle \mid \xi \perp V, \|\xi\| = 1 \} \quad (3.1)$$

be the j -th eigenvalue of $\mathcal{H}(x)$. Furthermore, we put

$$c(x) := \begin{cases} \|\mathcal{H}(x)^{-1/2} J(x) \mathcal{H}(x)^{-1/2}\|, & \det(\mathcal{H}(x)) \neq 0, \\ \infty, & \text{otherwise.} \end{cases} \quad (3.2)$$

We have estimates

$$\frac{1}{c(x)} \leq \|\mathcal{H}(x)^{1/2} J(x)^{-1} \mathcal{H}(x)^{1/2}\| \leq \|J(x)^{-1}\| \lambda_n(x), \quad (3.3)$$

and, if $\det \mathcal{H}(x) \neq 0$,

$$c(x) \leq \|J(x)\| \|\mathcal{H}(x)^{-1}\| = \frac{\|J(x)\|}{\lambda_1(x)}. \quad (3.4)$$

Thus we have for all $x \in \mathbb{R}$

$$\frac{\lambda_1(x)}{\|J(x)\|} \leq \frac{1}{c(x)} \leq \|J(x)^{-1}\| \lambda_n(x). \quad (3.5)$$

In view of (3.3) the function $\frac{1}{c(x)}$ is locally integrable. The significance of $c(x)$ stems from the fact that if $\det(\mathcal{H}(x)) \neq 0$ then for $\xi \in \mathbb{C}^n$ we have the estimate

$$\begin{aligned} \xi^* J(x)^* \mathcal{H}(x)^{-1} J(x) \xi &= \|\mathcal{H}(x)^{-1/2} J(x) \xi\|^2 \leq c(x)^2 \|\mathcal{H}(x)^{1/2} \xi\|^2 \\ &= c(x)^2 \xi^* \mathcal{H}(x) \xi. \end{aligned} \quad (3.6)$$

LEMMA 3.1. *Let $f \in L^1_{\text{loc}}(\mathbb{R})$, $f(x) \geq 0$, be a non-negative locally integrable function. Assume in addition that*

$$\int_0^\infty f(x) dx = +\infty.$$

Then for $n \in \mathbb{N}$ there exists an absolute continuous function $\chi_n \in \text{AC}(\mathbb{R})$ with the properties

- (1) $\chi_n(x) = 1, x \leq n$,
- (2) $\chi_n(x) = 0, x \geq x_n$, for some x_n ,
- (3) $\chi'_n \in L^\infty(\mathbb{R})$,
- (4) $|\chi'_n(x)| \leq \frac{1}{n} f(x)$, for all $x \in \mathbb{R}$.

PROOF. Fix $n \in \mathbb{N}$. By B. Levy's theorem on monotone convergence we have

$$\lim_{C \rightarrow +\infty} \int_0^\infty \frac{1}{n} \min(C, f(x)) dx = +\infty,$$

and thus we may choose $C > 0$ such that

$$\int_n^\infty \min(C, \frac{1}{n} f(x)) dx \geq 2.$$

Now choose N large enough such that

$$K_n := \int_n^N \min(C, \frac{1}{n} f(x)) dx \geq 1$$

and put

$$\chi_n(x) := 1 - \frac{1}{K_n} \int_{\min(n,x)}^{\min(N,x)} \min(C, \frac{1}{n} f(s)) ds.$$

χ_n has the desired properties with $x_n = N$. □

THEOREM 3.2. *Let $S_{\min} = S_{\min}(J, B, \mathcal{H})$ be a first order system (2.1) on the interval I .*

(1) *Let $I = \mathbb{R}$ and assume that*

$$\pm \int_0^{\pm\infty} \frac{1}{c(x)} dx = +\infty. \quad (3.7)$$

Then S_{\min} is essentially self-adjoint, i.e. $S = S_{\max}$.

(2) *Let $I = \mathbb{R}_+$ and assume that*

$$\int_0^\infty \frac{1}{c(x)} dx = +\infty. \quad (3.8)$$

Then for $\{\tilde{f}, \tilde{g}\} \in S_{\max}$ there exists a sequence $\{f_n, g_n\} \in \mathcal{S}_{\max}$ such that $\tilde{f}_n \rightarrow \tilde{f}, \tilde{g}_n \rightarrow \tilde{g}$ in $L^2_{\mathcal{H}}(\mathbb{R}_+)$ and $\text{supp } f_n, \text{supp } g_n \subset [0, \infty)$ compact. Moreover, for $\{\tilde{f}_j, \tilde{g}_j\} \in S_{\max}, j = 1, 2$ one has

$$\langle \tilde{f}_1, \tilde{g}_2 \rangle - \langle \tilde{g}_1, \tilde{f}_2 \rangle = -f_1(0)^* J f_2(0). \quad (3.9)$$

REMARK 3.3. Note that the condition (3.7) (resp. (3.8)) implies that for each $R > 0$ there exist subsets $K_{\pm} \subset \mathbb{R}_{\pm} \setminus [-R, R]$ (resp. $K_+ \subset \mathbb{R}_+ \setminus [0, R]$) of positive Lebesgue measure such that the Hamiltonian \mathcal{H} is positive definite on K_{\pm} (resp. K_+). In particular, the corresponding system S is definite on \mathbb{R} (resp. on \mathbb{R}_+).

PROOF. (1) According to Lemma 3.1 let χ_n be absolutely continuous with bounded derivative,

$$\chi_n(x) = \begin{cases} 1, & |x| \leq n, \\ 0, & |x| \geq x_n, \end{cases}$$

and

$$|\chi'_n(x)| \leq \frac{1}{nc(x)}.$$

For $\{\tilde{f}, \tilde{g}\} \in S_{\max}$ we choose, according to Proposition 2.12, representatives $\{f, g\} \in \mathcal{S}_{\max}$ and put

$$f_n := \chi_n f.$$

Since χ'_n vanishes if $\mathcal{H}(x)$ is not invertible the function $\chi'_n \mathcal{H}(x)^{-1} J(x) f$ is well-defined. Moreover

$$\begin{aligned} \|\chi'_n \mathcal{H}^{-1} J f\|_{L^2_{\mathcal{H}}(\mathbb{R})}^2 &\leq \int_{\mathbb{R}} |\chi'_n(x)|^2 f(x)^* J(x)^* \mathcal{H}(x)^{-1} J(x) f(x) dx \\ &\leq \sup_{x \in \mathbb{R}} (\chi'_n(x) c(x))^2 \|f\|_{L^2_{\mathcal{H}}(\mathbb{R})}^2 \\ &\leq \frac{1}{n^2} \|f\|_{L^2_{\mathcal{H}}(\mathbb{R})}^2, \end{aligned}$$

hence $\chi'_n \mathcal{H}(x)^{-1} J(x) f$ lies in $\mathcal{L}^2_{\mathcal{H}}(\mathbb{R})$ and it converges to 0 in $\mathcal{L}^2_{\mathcal{H}}(\mathbb{R})$. Finally, we calculate

$$\begin{aligned} Jf'_n + Bf_n &= \chi_n(Jf' + Bf) + \chi'_n Jf \\ &= \mathcal{H}(\chi_n g + \chi'_n \mathcal{H}^{-1} Jf) \\ &=: \mathcal{H} g_n. \end{aligned}$$

Thus $\{f_n, g_n\} \in \mathcal{S}$ and $\lim_{n \rightarrow \infty} \{\tilde{f}_n, \tilde{g}_n\} = \{\tilde{f}, \tilde{g}\}$ and the claim is proved.

The proof of (2) proceeds along the same lines with minor modifications.

(3.9) follows from integration by parts if f_2, g_2 have compact support. To prove it in general we consider $f_{2,n} = \chi_n f_2$ and $g_{2,n} = \chi_n g_2 + \chi'_n \mathcal{H}^{-1} J f_2$. Then (3.9) holds true for $\{\tilde{f}_1, \tilde{g}_1\}$ and $\{\tilde{f}_{2,n}, \tilde{g}_{2,n}\}$. Noting that $f_{2,n}(0) = f_2(0)$ is independent of n we obtain the result by taking the limit as $n \rightarrow \infty$. \square

REMARK 3.4. (3.7) is not necessary for S_{\min} to be essentially self-adjoint. Namely, in the situation of Example 2.17 2. we have $\frac{1}{c(x)} = 0$. But there certainly exist V (e.g. $V \in L^\infty(\mathbb{R})$) such that the Schrödinger operator $-\frac{d^2}{dx^2} + V$ and hence the Hamiltonian in Example 2.17 2. are essentially self-adjoint.

See also Example 5.33 for a counterexample with a nonsingular Hamiltonian \mathcal{H} .

COROLLARY 3.5. *If $J = J(0)$ is constant then the condition (3.7) (resp. (3.8)) is implied by*

$$\pm \int_0^{\pm\infty} \lambda_1(x) dx = \infty \quad (\text{resp. } \int_0^\infty \lambda_1(x) dx = \infty). \quad (3.10)$$

Hence (for $J = J(0)$) (3.10) implies the conclusions in Theorem 3.2.

PROOF. This follows immediately from the estimate (3.5). \square

REMARK 3.6. It is clear that S_{\min} is essentially self-adjoint iff for each $f \in \mathcal{D}(\mathcal{S}_{\max})$ the following limit exists:

$$\lim_{x \rightarrow \pm\infty} f(x)^* J(x) f(x) = 0. \quad (3.11)$$

Condition (3.10) yields a stronger conclusion about $\mathcal{D}(\mathcal{S}_{\max})$. In order to explain it we denote by $\text{AC}_0(\mathbb{R}, \mathbb{C}^n)$ the set of those $f \in \text{AC}(\mathbb{R}, \mathbb{C}^n)$ such that there exist sequences $x_n^\pm \rightarrow \pm\infty$ with $\lim_{n \rightarrow \infty} \langle f(x_n^\pm), f(x_n^\pm) \rangle = 0$.

It is clear that under condition (3.10) $\mathcal{D}(\mathcal{S}_{\max}) \subset \text{AC}_0(\mathbb{R}, \mathbb{C}^n)$. The converse assertion is also true if $B = 0$ and $\mathcal{H}(x) = \text{diag}(\lambda_1(x), \dots, \lambda_n(x))$ is a diagonal matrix with eigenvalues $\lambda_1(x) \leq \dots \leq \lambda_n(x)$. Indeed, if $\lambda_1 \in L^1(\mathbb{R})$ then

$$f := \text{col}(1, 0, \dots, 0) \in \mathcal{D}(\mathcal{S}_{\max}) \text{ but } \langle f(x), f(x) \rangle = 1 \text{ and } f \notin \text{AC}_0(\mathbb{R}, \mathbb{C}^n).$$

Probably (3.10) is equivalent to the inclusion $\mathcal{D}(\mathcal{S}_{\max}) \subset \text{AC}_0(\mathbb{R}, \mathbb{C}^n)$ for an arbitrary $S_{\max}(J, B, \mathcal{H})$ with constant $J = J(0)$.

However (3.10) is weaker than (3.7) as the following example shows:

EXAMPLE 3.7. Let

$$\mathcal{H}(x) = \text{diag}(\lambda_1(x), \lambda_2(x)), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where

$$\lambda_1(x) = (|x| + 2)^{-1} \ln^{-2}(|x| + 2), \quad \lambda_2(x) = (2 + |x|)^{-1}.$$

Then $\lambda_1(x) \in L^1(\mathbb{R})$, but

$$(\lambda_1 \lambda_2)^{1/2} \notin L^1(\mathbb{R}_\pm).$$

Since $\pm i(\lambda_1 \lambda_2)^{1/2}$ are the eigenvalues of $\mathcal{H}^{1/2} J^{-1} \mathcal{H}^{1/2}$ we infer that $\frac{1}{c(x)} = (\lambda_1(x) \lambda_2(x))^{1/2}$. Hence (3.7) is satisfied but $\lambda_1 \in L^1(\mathbb{R})$.

Besides, setting

$$f = \text{col}(\ln^{1/4}(2 + |x|), 0), \quad g = \text{col}(0, -4^{-1} \text{sgn}(x) \ln^{-3/4}(2 + |x|)) \in \mathcal{L}_{\mathcal{H}}^2(\mathbb{R})$$

one gets $\{f, g\} \in S_{\max}$ but $\langle f(x), f(x) \rangle = \ln^{1/2}(2 + |x|) \rightarrow \infty$ as $x \rightarrow \pm\infty$ and $f \notin \text{AC}_0(\mathbb{R}, \mathbb{C}^2)$.

3.2. The case of a symmetric operator. For completeness we briefly comment on the case that the system (2.1) defines a symmetric linear operator containing at least the C^1 -functions with compact support in its domain. Namely, let J, B, \mathcal{H} be as in (2.2) and assume in addition that $\mathcal{H}(x)$ is invertible for all $x \in I$ and that $\mathcal{H}(x)^{-1}, B^* \mathcal{H}^{-1} B$ is locally integrable. In this case each class $\tilde{f} \in L_{\mathcal{H}}^2(I)$ contains at most one continuous representative. In particular $\text{AC}_{\text{comp}}(I, \mathbb{C}^n)$ may be viewed as a subset of $L_{\mathcal{H}}^2(I)$.

Then we consider the differential operator

$$L := \mathcal{H}^{-1} \left(J \frac{d}{dx} + B \right). \tag{3.12}$$

L maps $\mathcal{D}(L) := C_{\text{comp}}^1(I, \mathbb{C}^n)$ into $L_{\mathcal{H}}^2(I)$. Namely, if $f \in C_{\text{comp}}^1(I, \mathbb{C}^n)$, $K := \text{supp}(f) \subset I$ then we estimate

$$\begin{aligned} \|\mathcal{H}^{-1} J f'\|_{\mathcal{H}}^2 &= \left| \int_K f'(x)^* J(x) \mathcal{H}(x)^{-1} J(x) f'(x) dx \right| \\ &\leq \sup_{x \in K} \|J(x) f'(x)\|^2 \int_K \|\mathcal{H}(x)^{-1}\| dx < \infty \\ \|\mathcal{H}^{-1} B f\|_{\mathcal{H}}^2 &= \int_K f(x)^* B(x)^* \mathcal{H}(x)^{-1} B(x) f(x) dx \\ &\leq \sup_{x \in K} \|f(x)\|^2 \int_K \|B(x)^* \mathcal{H}(x)^{-1} B(x)\| dx < \infty. \end{aligned} \tag{3.13}$$

L is formally symmetric and in view of the regularity Theorem 2.4 the domain $\mathcal{D}(L_{\max})$ of $L_{\max} := L^*$ lies in $\text{AC}(I, \mathbb{C}^n)$. Furthermore, for $f, g \in \mathcal{D}(L_{\max})$ and $\alpha < \beta$ we have

$$\begin{aligned} & \int_{\alpha}^{\beta} (L_{\max} f)(x)^* \mathcal{H}(x) g(x) dx - \int_{\alpha}^{\beta} f(x)^* \mathcal{H}(x) (L_{\max} g)(x) dx \\ &= -f(\beta)^* J(\beta) g(\beta) + f(\alpha)^* J(\alpha) g(\alpha). \end{aligned} \quad (3.14)$$

In contrast to general first order systems the domain of L_{\max} is localizable in the following sense: $C_0^\infty(I, \mathbb{C}^n)$ is dense in

$$\mathcal{D}_{\text{comp}}(L_{\max}) := \{f \in \mathcal{D}(L_{\max}) \mid \text{supp}(f) \text{ is compact}\}$$

with respect to the graph norm of L . Namely, from (3.14) we infer that for $f \in \mathcal{D}_{\text{comp}}(L_{\max}), g \in \mathcal{D}(L_{\max})$ we have $\langle L_{\max} f, g \rangle = \langle f, L_{\max} g \rangle$, i.e. $f \in \mathcal{D}(L_{\max}^*) = \mathcal{D}(L^{**}) = \mathcal{D}(\overline{L})$.

Summing up one arrives at the following result.

THEOREM 3.8. *Let $I = \mathbb{R}$ and let $\mathcal{H}(x)$ be invertible for $x \in \mathbb{R}$. Assume also that \mathcal{H}^{-1} and $B^* \mathcal{H}^{-1} B$ are locally integrable and (3.7) holds. Then the operator L is essentially self-adjoint on $C_0^\infty(\mathbb{R}, \mathbb{C}^n) \subset L_{\mathcal{H}}^2(\mathbb{R}, \mathbb{C}^n)$.*

COROLLARY 3.9. *In the framework of Theorem 3.8 assume that $J(x)$ is bounded on \mathbb{R} and that there exists a $\delta > 0$ such that $\mathcal{H}(x) \geq \delta > 0$ for $x \in \mathbb{R}$. Moreover, assume $B \in L_{\text{loc}}^2(\mathbb{R}, \text{M}(n, \mathbb{C}))$. Then L is essentially self-adjoint on $\text{AC}_{\text{comp}}(\mathbb{R}, \mathbb{C}^n) \subset L_{\mathcal{H}}^2(\mathbb{R}, \mathbb{C}^{2n})$.*

PROOF. $\mathcal{H}(x) \geq \delta > 0$ implies that $\mathcal{H}(x)^{-1/2} \leq \delta^{-1/2}$. Hence (3.7) holds since $\|\mathcal{H}(x)^{-1/2} J(x) \mathcal{H}(x)^{-1/2}\| \leq C \delta^{-1}$. Moreover, from $\mathcal{H}(x) \geq \delta > 0$ we infer that \mathcal{H}^{-1} is bounded and hence $B^* \mathcal{H}(x)^{-1} B \leq \frac{1}{\delta} B^* B \in L_{\text{loc}}^1$. Hence Theorem 3.8 applies. \square

COROLLARY 3.10. *In the framework of Theorem 3.8 let $J(x)$ be bounded on \mathbb{R} and let $\mathcal{H}(x) = T^*(x) \mathcal{H}_1(x) T(x)$ such that*

- (1) $T(x)$ and $\mathcal{H}_1(x)$ are continuous on \mathbb{R} ,
- (2) $\mathcal{H}_1(x) \geq \delta > 0$, $x \in \mathbb{R}$,
- (3) $T^*(x) Q(x) T(x) = J(x)$, where Q is continuous and bounded.
- (4) $B \in L_{\text{loc}}^2(\mathbb{R}, \text{M}(n, \mathbb{C}))$.

Then L is essentially self-adjoint on $\text{AC}_{\text{comp}}(\mathbb{R}, \mathbb{C}^n) \subset L_{\mathcal{H}}^2(\mathbb{R}, \mathbb{C}^{2n})$.

PROOF. Since $T^* Q T = J$ it is clear that $T(x)$ and $Q(x)$ are invertible for all x . Furthermore, $\mathcal{H}(x) = T^*(x) \mathcal{H}_1(x) T(x) \geq \delta T^*(x) T(x)$, hence $\mathcal{H}(x) > 0$ for all x . Setting $K(x) := \mathcal{H}^{-1/2}(x) T^*(x)$, one has $\|K(x)\| \leq \delta^{-1/2}$ and thus

$$\begin{aligned} c(x) &\leq \|\mathcal{H}^{-1/2}(x) J(x) \mathcal{H}^{-1/2}(x)\| \\ &\leq \|K(x) Q(x) K^*(x)\| \\ &\leq \delta^{-1} \|Q(x)\| \leq c \delta^{-1} \end{aligned}$$

since $Q(x)$ is bounded on \mathbb{R} . Hence (3.7) is fulfilled and we reach the conclusion. \square

REMARK 3.11. 1. If $\mathcal{H}(x)$ is invertible for almost all $x \in \mathbb{R}$ then by Theorem 3.2 the operator L_{\min} defined by (3.12) on

$$\mathcal{D}(L_{\min}) = \mathcal{D}(S_{\min}) = \{f \in \text{AC}_{\text{comp}}(\mathbb{R}, \mathbb{C}^n) \mid Lf \in L^2_{\mathcal{H}}(\mathbb{R})\}$$

is essentially self-adjoint under the only condition (3.7).

However, we cannot conclude the essential self-adjointness of L on $C_0^\infty(\mathbb{R}, \mathbb{C}^n)$ without additional assumptions (like in Theorem 3.8) since in general $C_{\text{comp}}^1(\mathbb{R}, \mathbb{C}^n)$ is not contained in $\mathcal{D}(L_{\min})$.

2. Corollary 3.9 and Corollary 3.10 have been obtained by L. Sakhnovich [28] under the additional assumptions $B = 0$ and $J = J(0)$ constant.

3. In [22, Proposition 2.1] we established self-adjointness of the operator L with $\mathcal{H} = I$ and $J(x) = J(0)$ being constant. This fact is well-known. It is contained, e.g., as a very special case in a result due to Levitan and Otelbaev [23, Theorem 2].

Note however that the proof of Proposition 2.1 from [22] remains valid if $J(x)$ is nonconstant and bounded on \mathbb{R} . Corollary 3.9 is reduced to this result via the gauge transformation (2.8) with $U = \mathcal{H}^{-1/2}$.

4. Essential self-adjointness on the line. Second approach.

In this section we present a second proof of the essential self-adjointness of the operator L from Subsection 3.2. This second proof uses the hyperbolic equation method (cf. [5],[9]).

If the coefficients of L are smooth then this method even proves the essential self-adjointness of all powers L^n ($n \in \mathbb{Z}_+$) of the operator L [9].

We recall some definitions and results. Let H be a densely defined operator in a Hilbert space \mathfrak{H} . Recall that a vector function $u : [0, \infty) \rightarrow \mathfrak{H}$ is called a strong solution of the equation

$$\frac{du}{dt}(t) + Hu(t) = 0, \quad t \in (0, \infty), \quad (4.1)$$

if u is strongly differentiable, $u(t) \in \mathcal{D}(H)$ for each $t \in (0, \infty)$ and (4.1) is satisfied for each $t \in (0, \infty)$.

Our second proof of the essential self-adjointness is based on the following result due to Berezanskii-Povzner (cf. also [9]).

THEOREM 4.1 ([5]). *Let H be a symmetric operator in a Hilbert space \mathfrak{H} . For the operator H to be essentially self-adjoint in \mathfrak{H} it is necessary and sufficient that for some $b > 0$ the function $u = 0$ is the only strong solution of the Cauchy problems*

$$\frac{du}{dt}(t) \pm (iH)^*u(t) = 0, \quad t \in [0, b], \quad u(0) = 0. \quad (4.2)$$

We return to the operator $L = \mathcal{H}^{-1}(J \frac{d}{dx} + B)$ from the previous Subsection 3.2.

For a real number α let $\alpha_\pm(t)$ be the unique solution of the initial value problem

$$y'(t) = \pm c(y(t)), \quad y(0) = \alpha. \quad (4.3)$$

Here, c is the function defined in (3.2). Note that c and $\frac{1}{c}$ are locally integrable and hence the first order equation (4.3) with separated variables has a unique solution. Moreover, if

$$\pm \int_0^{\pm\infty} \frac{1}{c(x)} dx = \infty \quad (4.4)$$

then the solutions $\alpha_{\pm}(t)$ exist for all $t \in \mathbb{R}$.

PROPOSITION 4.2 (Local energy estimate). *Let s_t be a strong solution of the equation*

$$\frac{d}{dt}u - iL^*u = 0 \quad (4.5)$$

defined for $|t| < \varepsilon$. Moreover, assume that for some $\alpha < \beta$ the functions $\alpha_{\pm}, \beta_{\pm}$ are defined for $|t| < \varepsilon$. Then the function

$$F_{\alpha,\beta}(t) := \int_{\alpha_+(t)}^{\beta_-(t)} s_t(x)^* \mathcal{H}(x) s_t(x) dx$$

is a decreasing function of t .

In particular, if $\text{supp}(s_0) \subset [\alpha, \beta]$ then $\text{supp}(s_t) \subset [\alpha_-(t), \beta_+(t)]$.

PROOF. Differentiation by t and integration by parts yields in view of (3.14)

$$\begin{aligned} & \frac{d}{dt} \int_{\alpha_+(t)}^{\beta_-(t)} s_t(x)^* \mathcal{H}(x) s_t(x) dx \\ &= -c(\beta_-(t))(s_t^* \mathcal{H} s_t)(\beta_-(t)) - c(\alpha_+(t))(s_t^* \mathcal{H} s_t)(\alpha_+(t)) \\ & \quad - i \int_{\alpha_+(t)}^{\beta_-(t)} (L^* s_t)(x)^* \mathcal{H}(x) s_t(x) - s_t(x)^* \mathcal{H}(x) (L^* s_t)(x) dx \\ &= -c(\beta_-(t))(s_t^* \mathcal{H} s_t)(\beta_-(t)) - c(\alpha_+(t))(s_t^* \mathcal{H} s_t)(\alpha_+(t)) \\ & \quad - i(s_t^* J^* s_t)(\beta_-(t)) + i(s_t^* J^* s_t)(\alpha_+(t)) \end{aligned} \quad (4.6)$$

and by definition of c this is ≤ 0 (cf. (3.6)). Note that all terms in (4.6) are real.

The last statement is clear. \square

PROPOSITION 4.3 (Local existence). *For each $f \in \mathcal{D}_{\text{comp}}(L^*)$ there exists an $\varepsilon > 0$ and a unique strong solution $s_t, |t| < \varepsilon$, of the equation (4.5) satisfying the initial condition $u(0) = f$. Moreover $s_t \in \mathcal{D}_{\text{comp}}(L^*)$ for all t .*

PROOF. Assume that $\text{supp}(f) \subset [-N, N]$ for some $N > 0$. Choose a self-adjoint extension, L_N , of L on the interval $[-2N, 2N]$. This is possible since in view of Proposition 2.21 the deficiency indices of L on the finite interval $[-2N, 2N]$ are given by $\mathcal{N}_{\pm}(L) = N_{\pm}(L) = n$.

Next let $s_t(x) := e^{itL_N} f$ be the strong solution of the wave equation for L_N . The local energy estimate above shows that for t small enough, s_t has compact support in $[-2N - \delta, 2N + \delta]$ and hence can be extended by 0 to a strong solution of the wave equation for L^* .

The uniqueness follows immediately from the local energy estimate. \square

Now we can give the

SECOND PROOF OF THEOREM 3.8. If (3.7) is fulfilled then the previous result shows that for each $f \in \mathcal{D}_{\text{comp}}(L^*)$ there exists a unique strong solution $s_t, t \in \mathbb{R}$, of the Cauchy problem for the wave equation (4.5) and $s_t \in \mathcal{D}_{\text{comp}}(L^*)$ for all t . Hence the result follows from Theorem 4.1. \square

5. Defect numbers and essential self-adjointness on the half-line

In this section we present some results on the square-integrable solutions of the system

$$J(x)y'(x) + B(x)y(x) = \lambda \mathcal{H}(x)y(x) \quad (5.1)$$

on the half lines \mathbb{R}_{\pm} . As in Section 2 we associate with equation (5.1) the minimal symmetric linear relations $\mathcal{S}_{\min, \pm}$ and $S_{\min, \pm}$ in $\mathcal{L}_{\mathcal{H}}^2(\mathbb{R}_{\pm}, \mathbb{C}^n)$ and $L_{\mathcal{H}}^2(\mathbb{R}_{\pm}, \mathbb{C}^n)$ respectively; $\mathcal{S}_{\pm}, S_{\pm}, \mathcal{S}_{\max, \pm}, S_{\max, \pm}$ are defined accordingly (cf. Def. 2.3). As in Section 2 we denote by $\mathcal{N}_{\pm}(\mathcal{S}_{\pm}) := \dim \mathcal{E}_{\pm i}(\mathcal{S}_{\pm})$ the formal deficiency indices of the system (5.1).

If in addition $\mathcal{H}(x)$ is invertible for almost all $x \in \mathbb{R}_{\pm}$ then \mathcal{S}_{\pm} is an operator. In this case the formal defect subspace $\mathcal{E}_{\lambda}(\mathcal{S}_{\pm})$ coincides with defect subspace $E_{\lambda}(S_{\pm})$ of the operator S_{\pm} .

We denote by $\kappa_{+} := \kappa_{+}(iJ(0))$ and $\kappa_{-} := \kappa_{-}(iJ(0))$ respectively the numbers of positive and negative eigenvalues of the matrix $iJ(0)$. Since $\det J(x) \neq 0$ for $x \in I$ it is clear that $\kappa_{\pm}(iJ(x))$ does not depend on $x \in I$, $\kappa_{\pm}(iJ(0)) = \kappa_{\pm}(iJ(x))$. In what follows we will write sometimes $\kappa_{\pm}(iJ)$ instead of $\kappa_{\pm}(iJ(0))$. Recall the well-known estimates (see [2, Theorem 9.11.1])

$$\kappa_{\pm} \leq \mathcal{N}_{\pm}(\mathcal{S}_{\pm}) \leq n, \quad (5.2a)$$

$$\kappa_{\pm} \leq \mathcal{N}_{\mp}(\mathcal{S}_{-}) \leq n, \quad (5.2b)$$

$$\mathcal{N}_{+}(\mathcal{S}_{\pm}) + \mathcal{N}_{-}(\mathcal{S}_{\pm}) \geq n. \quad (5.2c)$$

REMARK 5.1. These inequalities have been established in [2] by a generalization of the well-known Weyl analytic (circle-point) method. We note that in the case $\mathcal{N}_{+}(\mathcal{S}_{+}) = \mathcal{N}_{-}(\mathcal{S}_{+}), \kappa_{+} = \kappa_{-} = n/2$ they follow easily from the results of Subsection 2.5:

For simplicity let us assume that the system S_{+} is definite on \mathbb{R}_{+} . Then by Proposition 2.26 (2),(3) $\dim(\mathcal{S}_{+}^{*}/\mathcal{S}_{+}) \geq n$ and by (2.78) $\mathcal{N}_{+}(\mathcal{S}_{+}) + \mathcal{N}_{-}(\mathcal{S}_{+}) = \dim(\mathcal{S}_{+}^{*}/\mathcal{S}_{+}) \geq n$. If $\mathcal{N}_{+}(\mathcal{S}_{+}) = \mathcal{N}_{-}(\mathcal{S}_{+})$ then $\mathcal{N}_{\pm}(\mathcal{S}_{+}) \geq n/2$. These inequalities imply (5.2) if $\kappa_{+}(iJ) = \kappa_{-}(iJ) = n/2$.

However, we emphasize that we did not succeed to prove the estimates (5.2) in full generality in the framework of extension theory.

Finally, note that, e.g., if $J^{-1}\mathcal{H}$ is real then $\mathcal{N}_{+}(\mathcal{S}_{+}) = \mathcal{N}_{-}(\mathcal{S}_{+})$, cf. Proposition 5.23 below.

5.1. Minimal deficiency indices. Here we present a result on minimal possible deficiency (and formal deficiency) indices. It may be directly obtained by combining Theorem 3.2 and Proposition 5.4 below but in order to demonstrate "formal" approach we present a simple independent proof.

THEOREM 5.2. *Let $c(x)$ be the function defined in (3.2). If*

$$\int_0^\infty \frac{1}{c(x)} dx = \infty \quad \left(\text{resp. } \int_{-\infty}^0 \frac{1}{c(x)} dx = \infty \right) \quad (5.3)$$

then $\mathcal{N}_\pm(\mathcal{S}_+) = N_\pm(S_+) = \kappa_\pm$ (resp. $\mathcal{N}_\pm(\mathcal{S}_-) = N_\pm(S_-) = \kappa_\mp$).

PROOF. It suffices to prove the Theorem for the linear relation S_+ . As noted in Remark 3.3 it follows from (5.3) that S_+ is definite. Therefore by Proposition 2.26 (1) $\mathcal{N}_\pm(\mathcal{S}_+) = N_\pm(S_+)$. Thus it suffices to prove the assertions for $\mathcal{N}_\pm(\mathcal{S}_+)$.

Let y be a solution of (5.1) with $\lambda = \pm i$. Let $(a_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ be any sequence converging to ∞ . Then integrating by parts and taking (2.2) into account one gets

$$\begin{aligned} \|y\|_{\mathcal{H}}^2 &= \int_{\mathbb{R}_+} y(t)^* \mathcal{H}(t) y(t) dt \\ &= \lim_{k \rightarrow \infty} \left(-\lambda \int_0^{a_k} y^*(t) J(t) y'(t) dt - \lambda \int_0^{a_k} y^*(t) B(t) y(t) dt \right) \\ &= \lim_{k \rightarrow \infty} \left([-\lambda y^*(t) J(t) y(t)]_0^{a_k} - \lambda \int_0^{a_k} (J(t) y'(t))^* y(t) dt \right. \\ &\quad \left. + \lambda \int_0^{a_k} y^*(t) J'(t) y(t) dt - \lambda \int_0^{a_k} y^*(t) B(t) y(t) dt \right) \\ &= \lim_{k \rightarrow \infty} \left([-\lambda y^*(t) J(t) y(t)]_0^{a_k} - \int_0^{a_k} y^*(t) \mathcal{H}(t) y(t) dt \right). \end{aligned} \quad (5.4)$$

Thus $\lim_{k \rightarrow \infty} y(a_k)^* J(a_k) y(a_k)$ exists and

$$2\|y\|_{\mathcal{H}}^2 = -\lambda \lim_{k \rightarrow \infty} [y^*(t) J(t) y(t)]_0^{a_k}. \quad (5.5)$$

On the other hand we find using (3.2)

$$|y^*(t) J(t) y(t)| \leq c(t) \|\mathcal{H}(t)^{1/2} y(t)\|^2. \quad (5.6)$$

We claim that there is a sequence $(a_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{k \rightarrow \infty} c(a_k) \|\mathcal{H}(a_k)^{1/2} y(a_k)\|^2 = 0$. For if this were not the case then we had an estimate $c(x) \|\mathcal{H}(x)^{1/2} y(x)\|^2 \geq \delta > 0$ for $x \geq x_0$. This would contradict (5.3) and $\int_0^\infty \|\mathcal{H}(x)^{1/2} y(x)\|^2 dx = \|y\|_{\mathcal{H}}^2 < \infty$.

In view of (5.6) we have

$$\lim_{k \rightarrow \infty} \langle y(a_k), J(a_k) y(a_k) \rangle = 0. \quad (5.7)$$

Combining (5.5) and (5.7) one gets

$$2\|y\|_{\mathcal{H}}^2 = \langle \lambda J(0) y(0), y(0) \rangle_{\mathbb{C}^n}. \quad (5.8)$$

By the uniqueness theorem for first order differential equations the map $j : y(t) \rightarrow y(0)$ is an embedding of $\mathcal{E}_\pm(\mathcal{S}_+)$ into \mathbb{C}^n . Moreover, the quadratic form $\langle iJ(0)\xi, \xi \rangle$ is positive (resp. negative) on $j(\mathcal{E}_+(\mathcal{S}_+))$ (resp. $j(\mathcal{E}_-(\mathcal{S}_-))$). Since $\kappa_\pm(iJ)$ is just the number of positive (resp. negative) eigenvalues of the quadratic form $\langle iJ(0)\xi, \xi \rangle$ we obtain $\mathcal{N}_\pm(\mathcal{S}_\pm) \leq \kappa_\pm$. On the other hand we have in view of (5.2c)

$$n \leq \mathcal{N}_+(\mathcal{S}_+) + \mathcal{N}_-(\mathcal{S}_+) \leq \kappa_+ + \kappa_- = n \quad (5.9)$$

and thus equality holds. We emphasize that although we did not prove (5.2) in full generality the relation (5.2c) was proved completely in Remark 5.1. \square

COROLLARY 5.3. *Let $\lambda_1(x)$ be the smallest eigenvalue of $\mathcal{H}(x)$. If for some $a \geq 0$*

$$\int_a^\infty \lambda_1(x) dx = \infty \quad \left(\int_{-\infty}^{-a} \lambda_1(x) dx = \infty \right) \quad (5.10)$$

then $\mathcal{N}_\pm(\mathcal{S}_+) = N_\pm(S_+) = \kappa_\pm$ (resp. $\mathcal{N}_\pm(\mathcal{S}_-) = N_\pm(S_-) = \kappa_\mp$).

PROOF. This follows immediately from Corollary 3.5. \square

PROPOSITION 5.4. *Assume that the system (5.1) is definite on \mathbb{R}_+ and \mathbb{R}_- . Denote by $\mathcal{S}, \mathcal{S}_+, \mathcal{S}_-, S, S_+$, and S_- the symmetric linear relations associated to the equation (5.1) in $\mathcal{L}^2_{\mathcal{H}}(\mathbb{R}), \mathcal{L}^2_{\mathcal{H}}(\mathbb{R}_\pm), L^2_{\mathcal{H}}(\mathbb{R}), L^2_{\mathcal{H}}(\mathbb{R}_\pm)$ respectively. Then*

$$N_\pm(S) = N_\pm(S_+) + N_\pm(S_-) - n, \quad (5.11a)$$

$$\mathcal{N}_\pm(\mathcal{S}) = \mathcal{N}_\pm(\mathcal{S}_+) + \mathcal{N}_\pm(\mathcal{S}_-) - n. \quad (5.11b)$$

PROOF. It follows from definiteness and Proposition 2.26 (1) that (5.11a) and (5.11b) are equivalent. Hence it suffices to prove one of them.

We put $\mathcal{S}_0 := \mathcal{S}_+ \oplus \mathcal{S}_-$ and $S_0 := S_+ \oplus S_-$. By Proposition 2.26 (3) we have $f(0) = 0$ for each $f \in \mathcal{D}(\mathcal{S}_0)$. Moreover, Proposition 2.26 (2) implies that for each $\xi \in \mathbb{C}^n$ there exists $\{f, g\} \in \mathcal{S}$ with compact support such that $f(0) = \xi$. Hence $\dim(\mathcal{S}/\mathcal{S}_0) = n$. In view of Proposition 2.15 and Remark 2.16 the same argument applies to S_0 and S . Hence $\dim(S/S_0) = n$.

On the other hand since S is a closed symmetric extension of S_0 it follows from the second von Neumann formula (2.81) with \tilde{S} and S replaced by S and S_0 respectively, that $N_\pm(S) = N_\pm(S_0) - \dim(S/S_0) = N_\pm(S_0) - n$. Combining this formula with the obvious equalities $N_\pm(S_0) = N_\pm(S_+) + N_\pm(S_-)$ we obtain (5.11a) and thus also (5.11b). \square

REMARK 5.5. 1. The proof of the Proposition 5.4 is based essentially on the equality $\dim(\mathcal{S}/\mathcal{S}_0) = n$ which is a consequence of Proposition 2.26. Note however that if \mathcal{H} is positive definite on $[-a, a]$ ($a > 0$) then this fact is obvious. Namely, $\{f_j, g_j\}_1^n$ forms a basis of $\mathcal{S}(\text{mod } \mathcal{S}_0)$ if $f_j \in \text{AC}([-a, a], \mathbb{C}^n)$, $\text{supp } f_j \subset [-a, a]$, $f_j(0) = \{\delta_{kj}\}_{k=1}^n$ and $g_j := \chi \mathcal{H}^{-1}(Jf'_j + Bf_j)$, $j \in \{1, \dots, n\}$. Here, χ is a suitable cut-off function with support in $[-a, a]$ and $\chi|_{\text{supp } f_j} = 1$.

2. (5.11b) is due to Kogan and Rofo-Beketov [18, Theorem 2.3]. Their proof is analytical in character and close to that given by Bennewitz [4] for a similar formula

for the scalar equation $Su = \lambda Tu$, when one of the operators S, T has a strictly positive Dirichlet integral on the solutions.

Our proof, being operator-theoretic in character, is rather simple and follows that of Glazman's result on ordinary differential equations on the line ([1], [26]).

3. Proposition 5.4 leads to a simple relation between Theorem 3.2 and Theorem 5.2. Indeed combining (5.11), (5.2) and the obvious relation $\kappa_+ + \kappa_- = n$ we obtain the equivalences

$$\begin{aligned} N_+(S) = 0 &\iff N_+(S_\pm) = \kappa_\pm, \\ N_-(S) = 0 &\iff N_-(S_\pm) = \kappa_\mp. \end{aligned}$$

Thus Theorem 3.2 and Theorem 5.2 may be easily derived one from another.

4. (5.11) may be wrong for non-definite systems. For example let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = 0, \quad \mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (5.12)$$

One immediately checks that

$$\begin{aligned} N_+(S_\pm) = N_+(S) = N_-(S_\pm) = N_-(S) &= 0, \\ \mathcal{N}_+(S_\pm) = \mathcal{N}_+(S) = \mathcal{N}_-(S_\pm) = \mathcal{N}_-(S) &= 1. \end{aligned} \quad (5.13)$$

Consequently, neither (5.11a) nor (5.11b) holds.

5.2. The case of singular Hamiltonian. Next we want to present a criterion for the deficiency indices to be minimal on the half line \mathbb{R}_+ (for essential self-adjointness on the line \mathbb{R}) in a case where the Hamiltonian is singular everywhere.

We consider the type of first order systems introduced in Example 2.5 and thereafter. More precisely, we consider the first order system

$$\widetilde{J}f' + \widetilde{B}f = \widetilde{\mathcal{H}}g, \quad (5.14)$$

where

$$\widetilde{J} = \begin{pmatrix} 0 & J^* \\ -J & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} V & B \\ B^* - J' & -A \end{pmatrix}, \quad \widetilde{\mathcal{H}} = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & 0 \end{pmatrix}. \quad (5.15)$$

J, V, A, B, \mathcal{H} are assumed to satisfy the same assumptions as in (2.27). In addition, A is assumed to be non-negative. Theorem 5.2 does not apply to this situation since $\widetilde{\mathcal{H}}$ is singular at every point. It is clear that $L_{\widetilde{\mathcal{H}}}^2(I)$ is canonically isomorphic to $L_{\mathcal{H}}^2(I)$.

We put (cf. (2.6)) $\widetilde{\mathcal{S}}_+ = \mathcal{S}_+(\widetilde{J}, \widetilde{B}, \widetilde{\mathcal{H}})$. For simplicity we will consider the interval \mathbb{R}_+ only. For a function $f \in \mathcal{L}_{\widetilde{\mathcal{H}}}^2(\mathbb{R})$ we denote by f_1, f_2 the first resp. last n components.

We will use several times that if $\mathcal{H}(x)$ and $A(x)$ are invertible then we can estimate, for $\xi, \eta \in \mathbb{C}^n$,

$$\begin{aligned} |\xi^* J(x) \eta| &\leq \|A(x)^{1/2} \xi\| \|A(x)^{-1/2} J(x) \mathcal{H}(x)^{-1/2} \mathcal{H}(x)^{1/2} \eta\| \\ &\leq \|A(x)^{-1/2} J(x) \mathcal{H}(x)^{-1/2}\| \|A(x)^{1/2} \xi\| \|\mathcal{H}(x)^{1/2} \eta\|. \end{aligned} \quad (5.16)$$

Thus we put

$$c(x) := \begin{cases} \max(1, \|A(x)^{-1/2}J(x)\mathcal{H}(x)^{-1/2}\|), & \det(A(x)\mathcal{H}(x)) \neq 0, \\ \infty, & \text{otherwise.} \end{cases} \quad (5.17)$$

The self-adjointness criterion we are going to present will depend also on V . We assume that there exists an absolute continuous function $q \geq \delta > 0$ on \mathbb{R} such that

$$V \geq -q\mathcal{H}. \quad (5.18)$$

THEOREM 5.6. *Let $A(x)$ be positive semi-definite for each $x \in \mathbb{R}_+$ and let $c(x)$ be the function defined in (5.17). Let $q \geq \delta > 0$ be a function on \mathbb{R}_+ such that $V \geq -q\mathcal{H}$ and*

$$\int_0^\infty \frac{1}{c(x)q^{1/2}(x)}dx = \infty \quad \left(\int_{-\infty}^0 \frac{1}{c(x)q^{1/2}(x)}dx = \infty \right). \quad (5.19)$$

Moreover, assume that one of the following two conditions is satisfied:

(1) q is absolutely continuous and

$$\left| \frac{d}{dx} q^{-1/2}(x) \right| c(x) \leq C_1 \quad \text{for } x \in \mathbb{R}_+;$$

(2) $q(x)$ is non-decreasing (non-increasing).

Then $\mathcal{N}_\pm(\widetilde{\mathcal{S}}_+) = N_\pm(\widetilde{S}_+) = n$ ($\mathcal{N}_\pm(\widetilde{\mathcal{S}}_-) = N_\pm(\widetilde{S}_-) = n$).

PROOF. The set $\{x \in \mathbb{R} \mid \det(A(x)\mathcal{H}(x)) \neq 0\}$ has positive Lebesgue measure in view of (5.17) and (5.19). Therefore by Proposition 2.18. the system is definite. Hence it suffices to consider the formal deficiency indices.

1. Let y be a solution of (5.14) with $\lambda = \pm i$. We show that $y_2 \in \mathcal{L}_{q^{-1}A}^2(\mathbb{R}_+)$. (5.14) reads

$$\begin{aligned} J^*y_2' + Vy_1 + By_2 &= \lambda\mathcal{H}y_1, \\ Jy_1' - B^*y_1 + J'y_1 + Ay_2 &= 0. \end{aligned} \quad (5.20)$$

It follows that

$$\begin{aligned} \langle J^*y_2', y_1 \rangle + \langle Vy_1, y_1 \rangle + \langle By_2, y_1 \rangle &= \lambda \langle \mathcal{H}y_1, y_1 \rangle, \\ \langle y_2, Jy_1' \rangle + \langle y_2, J'y_1 \rangle - \langle By_2, y_1 \rangle + \langle y_2, Ay_2 \rangle &= 0. \end{aligned} \quad (5.21)$$

Adding (5.21) and integrating from 0 to x one gets

$$\begin{aligned} F(x)^2 &:= \int_0^x q(t)^{-1}y_2^*(t)A(t)y_2(t)dt = - \int_0^x q(t)^{-1}\langle y_2(t), J(t)y_1(t) \rangle' dt \\ &\quad - \int_0^x q(t)^{-1}y_1^*(t)V(t)y_1(t)dt + \lambda \int_0^x q(t)^{-1}y_1^*(t)\mathcal{H}(t)y_1(t)dt. \end{aligned} \quad (5.22)$$

We put $C_2 = q(0)^{-1}|\operatorname{Re}\langle y_2(0), J(0)y_1(0) \rangle|$ and recall (cf. (5.16)) that

$$c^{-1}(x)|\langle y_2(x), J(x)y_1(x) \rangle| \leq \|\mathcal{H}(x)^{1/2}y_1(x)\| \cdot \|A(x)^{1/2}y_2(x)\|. \quad (5.23)$$

Using this and the inequality $|q(x)^{-3/2}q'(x)c(x)| \leq C_1$ we obtain

$$\begin{aligned} \left| \int_0^x \left(\frac{1}{q(t)} \right)' \langle y_2(t), J(t)y_1(t) \rangle dt \right| &\leq \int_0^x \frac{q'(t)}{q^2(t)} c(t) \|\mathcal{H}^{1/2}(t)y_1(t)\| \cdot \|A^{1/2}(t)y_2(t)\| dt \\ &\leq C_1 \left(\int_0^x \|\mathcal{H}^{1/2}(t)y_1(t)\|^2 dt \right)^{1/2} \cdot \left(\int_0^x q(t)^{-1} \|A(t)^{1/2}y_2(t)\|^2 dt \right)^{1/2} \\ &\leq 2^{-1}C_1^2 \|y_1\|_{\mathcal{H}}^2 + 2^{-1}F^2(x). \end{aligned} \quad (5.24)$$

For brevity we assume in the sequel that $\delta = 1$ that is $q(x) \geq 1$. Now combining (5.23) and (5.24) and integrating by parts we have

$$\begin{aligned} \left| \operatorname{Re} \int_0^x q(t)^{-1} \langle y_2(t), J(t)y_1(t) \rangle' dt \right| &\leq C_2 + c(x) \|\mathcal{H}^{1/2}y_1(x)\| \cdot \|A^{1/2}y_2(x)\| \\ &\quad + 2^{-1}C_1^2 \|y_1\|_{\mathcal{H}}^2 + 2^{-1}F^2(x). \end{aligned} \quad (5.25)$$

Furthermore, the assumption $V \geq -q\mathcal{H}$ yields $-\int_0^x q^{-1}y_1^*V y_1 dt \leq \int_0^x y_1^*\mathcal{H} y_1 dt \leq \|y_1\|_{\mathcal{H}}^2$. Thus setting $C_3 := C_2 + (2^{-1}C_1^2 + 1) \cdot \|y_1\|_{\mathcal{H}}^2$ we infer from (5.22) and (5.25) that

$$\begin{aligned} \int_0^x \frac{F^2(t)dt}{2c(t)q(t)^{1/2}} &\leq \int_0^x \frac{C_3 dt}{c(t)q(t)^{1/2}} + \int_0^x \frac{1}{q(t)^{1/2}} \|\mathcal{H}^{1/2}y_1(t)\| \cdot \|A^{1/2}y_2(t)\| dt \\ &\leq \int_0^x \frac{C_3 dt}{c(t)q(t)^{1/2}} + \|y_1\|_{\mathcal{H}} \cdot F(x). \end{aligned} \quad (5.26)$$

We rewrite the latter inequality as

$$G(x) := \int_0^x 2^{-1}c^{-1}(t)q(t)^{-1/2}[F^2(t) - 2C_3]dt \leq \|y_1\|_{\mathcal{H}}F(x), \quad (5.27)$$

or as

$$G^2(x) \leq \|y_1\|_{\mathcal{H}}^2 (2c(x)q^{1/2}(x)G'(x) + 2C_3). \quad (5.28)$$

We claim that $F^2(t) \leq 2C_3$ for $t \in \mathbb{R}_+$. Assuming the contrary one finds x_0 such that $F(x_0) - 2C_3 =: \delta_1 > 0$, hence $F(x) - 2C_3 \geq \delta_1$ for $x \geq x_0$ since F is non-decreasing. Therefore in view of condition (5.19) $\lim_{x \rightarrow \infty} G(x) = \infty$.

On the other hand choosing $a \in \mathbb{R}_+$ such that $G(a) \geq 2C_3^{1/2}\|y_1\|_{\mathcal{H}}$, one derives from (5.28)

$$\begin{aligned} \frac{1}{2} \int_a^x \frac{dt}{c(t)q^{1/2}(t)} &\leq \int_a^x \frac{1}{c(t)q(t)^{1/2}} [1 - 2G(t)^{-2}C_3\|y_1\|_{\mathcal{H}}^2] dt \\ &\leq \int_a^x \frac{2G'(t)}{G^2(t)} dt = 2G(a)^{-1} - 2G(x)^{-1} \leq 2G(a)^{-1}. \end{aligned}$$

This inequality contradicts the condition (5.19). Thus $q^{-1/2}y_2 \in \mathcal{L}_A^2(\mathbb{R}_+)$ and $\|q^{-1/2}y_2\|_A^2 \leq 2C_3$.

2. Next we estimate using (5.16)

$$\begin{aligned}
|\langle y(x), \tilde{J}(x)y(x) \rangle| &\leq 2|\langle y_2(x), J(x)y_1(x) \rangle| \\
&\leq c(x)\|A(x)^{1/2}y_2(x)\|\|\mathcal{H}(x)^{1/2}y_1(x)\| \\
&\leq c(x)q(x)^{1/2}\|q(x)^{-1/2}A(x)^{1/2}y_2(x)\|\|\mathcal{H}(x)y_1(x)\|.
\end{aligned} \tag{5.29}$$

By 1. and Cauchy-Schwarz we know that $\|q(x)^{-1/2}A(x)^{1/2}y_2(x)\|\|\mathcal{H}(x)y_1(x)\|$ is integrable. In view of the condition (5.19) we infer exactly as in the proof of Theorem 5.2 that there is a sequence $(a_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ such that $\lim_{k \rightarrow \infty} |\langle y(a_k), \tilde{J}(a_k)y(a_k) \rangle| = 0$. Also as in the proof of Theorem 5.2 one now completes the proof, noting that $\kappa_{\pm}(\tilde{J}) = n$.

3. Now assume that condition (2) is satisfied. We reduce this case to the previous one. For this purpose it suffices to construct an absolutely continuous function \tilde{q} such that $\tilde{q}(x) \geq q(x)$ for $x \geq 0$ and \tilde{q} satisfies both (5.19) and (1).

Since $c^{-1}(x) \leq \|\mathcal{H}(x)^{1/2}J^{-1}(x)A(x)^{1/2}\| \leq \|\mathcal{H}(x)^{1/2}\| \cdot \|A(x)^{1/2}\| \cdot \|J^{-1}(x)\|$, one gets that $c^{-1} \in L^1_{\text{loc}}(\mathbb{R}_+)$. Therefore the function

$$t := \varphi(x) := \int_0^x c(s)^{-1} ds \tag{5.30}$$

is absolutely continuous and monotone increasing for $x > 0$. Denote by ψ the corresponding distribution function, $\psi(t) := \text{mes}\{x \in \mathbb{R}_+ \mid \varphi(x) \leq t\}$.

Next we put $q_1 := q \circ \psi$ and observe that q_1 is monotone increasing because so are q and ψ . Besides it is clear that

$$\int_0^\infty q_1(t)^{-1/2} dt = \int_0^\infty q(x)^{-1/2} c(x)^{-1} dx = \infty.$$

Following F. S. Rofo-Beketov [30] (see also [29]) one puts $\tilde{q}_1(n) = q_1(n+1)$ for $n \in \mathbb{Z}_+$ and then extends $\tilde{q}_1^{-1/2}$ to the semi-axis \mathbb{R}_+ by linear interpolation:

$$\tilde{q}_1(\lambda n + (1-\lambda)(n+1))^{-1/2} = \lambda \tilde{q}_1(n)^{-1/2} + (1-\lambda) \tilde{q}_1(n+1)^{-1/2}, \quad \lambda \in [0, 1].$$

It is clear that $\tilde{q}_1(x) \geq q_1(x)$ for $x \geq 0$. Moreover $\tilde{q}_1^{-1/2}$ is globally Lipschitz,

$$\left| \frac{d}{dx} \tilde{q}_1(x)^{-1/2} \right| \leq C_1 := q^{-1/2}(\psi(0)) \quad \text{and} \quad \int_{\mathbb{R}_+} \tilde{q}_1(t)^{-1/2} dt = \infty.$$

Finally, we put $\tilde{q} := \tilde{q}_1 \circ \varphi$ and check that \tilde{q} has the desired properties.

Indeed, $\tilde{q}(x) = \tilde{q}_1(\varphi(x)) \geq q_1(\varphi(x)) = q(\psi(\varphi(x))) \geq q(x)$, since $\psi(\varphi(x)) \geq x$, and therefore $V \geq -\tilde{q}\mathcal{H}$. Further, $\tilde{q}^{-1/2}$ is absolutely continuous because so is φ and $\tilde{q}_1^{-1/2}$ is Lipschitz. Now it follows from (5.30) that

$$|(\tilde{q}^{-1/2}(x))'| = |(\tilde{q}_1^{-1/2})'(\varphi(x))| \cdot \varphi'(x) \leq C_1 c(x)^{-1}$$

and

$$\int_{\mathbb{R}_+} \tilde{q}(x)^{-1/2} c(x)^{-1} dx = \int_{\mathbb{R}_+} \tilde{q}_1(t)^{-1/2} dt = \infty,$$

which completes the proof. \square

Combining Theorem 5.6 with Proposition 5.4 one arrives at the following self-adjointness criterion on the line.

THEOREM 5.7. *Let $\tilde{J}, \tilde{B}, \tilde{\mathcal{H}}$ be as in (5.15) with $A \geq 0$. Let $q \geq \delta > 0$ be a function on \mathbb{R} such that $V \geq -q\mathcal{H}$ and*

$$\pm \int_0^{\pm\infty} \frac{1}{c(x)q^{1/2}(x)} dx = \infty. \quad (5.31)$$

Moreover, assume that one of the following two conditions is satisfied:

(1) *q is absolutely continuous and*

$$\left| \frac{d}{dx} q^{-1/2}(x) \right| c(x) \leq C_1 \quad \text{for } x \in \mathbb{R};$$

(2) *$q(x)$ is non-increasing on \mathbb{R}_- and is non-decreasing on \mathbb{R}_+ .*

Then $\widetilde{S_{\min}} = S_{\min}(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$ is essentially self-adjoint.

REMARK 5.8. 1. Let $V \geq 0$. In this case the proof of Theorem 5.6 essentially simplified and one easily gets that $y_2 \in \mathcal{L}_A^2(\mathbb{R}_+)$ and $\|y_2\|_A^2 \leq |\operatorname{Re}\langle J(0)y_2(0), y_1(0) \rangle|$.

Moreover if $V \geq 0$ we may choose $q = 1$. Then Theorem 5.6 holds under the only condition $\pm \int_0^{\pm\infty} \frac{1}{c(x)} dx = +\infty$.

2. The condition (5.31) is satisfied if $\|J(x)\|$ is bounded and

$$\pm \int_0^{\pm\infty} \frac{\sqrt{\lambda_1(A(x))\lambda_1(\mathcal{H}(x))}}{q^{1/2}(x)} dx = \infty. \quad (5.32)$$

We apply Theorem 5.6 to the investigation of weighted matrix Sturm–Liouville (quasi-differential) equations with non-negative possibly singular (on some subsets of positive Lebesgue measure) weight \mathcal{H}

$$Py := -\frac{d}{dx}(A(x)^{-1}\frac{dy}{dx} + Q(x)y) + Q^*(x)\frac{dy}{dx} + R(x)y = \lambda\mathcal{H}(x)y, \quad (5.33)$$

where we A, Q, R, \mathcal{H} satisfy the same assumptions as in Example 2.5.

Denote by $\mathcal{N}_{\pm}(P_+)$ the formal deficiency indices of the equation (5.33) considered on the semiaxes \mathbb{R}_+ , that is the number of linearly independent solutions of (5.33) (with $\lambda \in \mathbb{C}_{\pm}$) belonging to $\mathcal{L}_{\mathcal{H}}^2(\mathbb{R}_+)$. By Proposition 2.20 the definition is correct, i.e. it does not depend on $\pm\lambda \in \mathbb{C}_+$.

THEOREM 5.9. *Let $P_+y = \lambda\mathcal{H}y$ be the equation of the form (5.33) with $A(x)$ being positive definite for $x \in \mathbb{R}_+$, $\mathcal{H} \geq 0$ and $c(x)$ be defined by (5.17) with $J = iI$. Suppose also that $V := R - Q^*AQ \geq -q\mathcal{H}$ where $q \geq \delta > 0$ and*

$$\int_0^{\infty} \frac{1}{c(x)q^{1/2}(x)} dx = \infty \quad \left(\int_{-\infty}^0 \frac{1}{c(x)q^{1/2}(x)} dx = \infty \right).$$

Moreover, assume that one of the following two conditions is satisfied:

(1) *$q^{-1/2}$ is absolutely continuous and*

$$\left| \frac{d}{dx} q^{-1/2}(x) \right| c(x) \leq C_1 \quad \text{for } x \in \mathbb{R}_+;$$

(2) *$q(x)$ is monotone increasing (monotone decreasing).*

Then $\mathcal{N}_\pm(P_+) = N_\pm(P_+) = n$ ($\mathcal{N}_\pm(P_-) = N_\pm(P_-) = n$).

PROOF. As elaborated in Example 2.5 the system (5.33) can be transformed into the first order system (5.14) $S(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$ with $\tilde{J}, \tilde{B}, \tilde{\mathcal{H}}$ defined in (2.18).

Namely, putting $u := y$ and $v := i(A^{-1}y' + Qy)$, one reduces the equation (5.33) to the system

$$\begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}' + \begin{pmatrix} R - Q^*AQ & -iQ^*A \\ iAQ & -A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} \mathcal{H} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \quad (5.34)$$

Since the corresponding linear relations are unitary equivalent, we apply Theorem 5.6 and reach the conclusion. \square

COROLLARY 5.10. *Let S_{\min} be the symmetric linear relation in $L^2_{\mathcal{H}}(\mathbb{R})$ induced by the Sturm–Liouville type (quasi-differential) equation (2.16). That is, $A, Q, R, \mathcal{H} \in L^1_{\text{loc}}(I, M(n, \mathbb{C}))$, $A(x)$ is positive definite for all $x \in \mathbb{R}$, and $\mathcal{H}(x) \geq 0$. Let $c(x)$ be as defined in (5.17). Suppose that $V := R - Q^*AQ \geq -q\mathcal{H}$, where $q \geq \delta > 0$ and*

$$\pm \int_0^{\pm\infty} \frac{1}{c(x)q^{1/2}(x)} dx = \infty.$$

Let also one of the following two conditions is satisfied:

- (1) $q^{-1/2}$ is absolutely continuous and $\left| \frac{d}{dx} q^{-1/2}(x) \right| c(x) \leq C_1$ for $x \in \mathbb{R}_+$;
- (2) $q(x)$ is monotone increasing on \mathbb{R}_+ and is monotone decreasing on \mathbb{R}_- .

Then S_{\min} is essentially self-adjoint.

PROOF. This follows immediately from Theorem 5.7 and Example 2.5. \square

REMARK 5.11. 1. Another reduction of the equation (5.33) to the first order system has been used in [11] for the investigation of the asymptotic behavior of eigenvalues of boundary value problems for the equation (5.33).

2. Theorem 5.9 generalizes some known results. Namely, for $Q = 0$, $A = \mathcal{H} = I_n$ and real R it has been obtained by V. B. Lidskii [24]. In turn for $n = 1$ Lidskii's result coincides with the well-known Titchmarsh–Sears theorem (see [6]).

On the other hand, if $n = 1$, $Q = 0$, $A = I_n$ and $R \geq 0$ the statement of Theorem 5.9 has been established by M. G. Krein [19] (see also [16]). In Remark 5.44 below we will discuss also Krein's result for R semibounded below ($R \geq -c \cdot I_n$, $c > 0$).

5.3. Maximal deficiency indices. Here we investigate the opposite case of maximal deficiency indices.

PROPOSITION 5.12. *Let $S_+ = S_+(J, 0, \mathcal{H})$ be a canonical system ($B = 0$) with a Hamiltonian $\mathcal{H}(x) = (h_{ij}(x))_{i,j=1}^n$ of positive type. If*

$$\int_0^\infty h_{jj}(x) dx < \infty, \quad \text{for } j = 1, \dots, k \quad (5.35)$$

then $\mathcal{N}_\pm(S_+) = N_\pm(S_+) \geq \max\{\kappa_\pm, k\}$.

PROOF. The condition $h_{jj} \in L^1(\mathbb{R}_+)$ is equivalent to the fact that the constant vector $u_j := \{\delta_{pj}\}_{p=1}^n$ is in $\mathcal{L}_{\mathcal{H}}^2(\mathbb{R}_+, \mathbb{C}^n)$. Thus $\mathcal{E}_0(\mathcal{S}_+) \supset \text{span}\{u_j \mid 1 \leq j \leq k\}$ and $\dim \mathcal{E}_0(\mathcal{S}_+) \geq k$. Since \mathcal{H} is of positive type the canonical system S_+ is definite. Therefore by Proposition 2.26 (1) we have $\mathcal{N}_{\pm}(\mathcal{S}_+) = N_{\pm}(S_+)$ and $\dim E_0(S_+) = \dim \mathcal{E}_0(\mathcal{S}_+) \geq k$. Now Proposition 2.26 (7) implies the assertion. \square

COROLLARY 5.13. *Let $S_+ = S_+(J, B, \mathcal{H})$ be a definite system. Let $Y(x) = Y(x, 0)$ be the fundamental matrix solution of the equation (5.1) (cf. (2.11)) and put $\widetilde{\mathcal{H}}(x) := Y^*(x)\mathcal{H}(x)Y(x) = (\widetilde{h}_{ij}(x))_{i,j=1}^n$. If the condition (5.35) is satisfied with h_{jj} replaced by \widetilde{h}_{jj} , then $\mathcal{N}_{\pm}(\mathcal{S}_+) = N_{\pm}(S_+) \geq k$.*

PROOF. The gauge transformation Y transforms the system into a canonical one with Hamiltonian \mathcal{H} and $\widetilde{B} = 0$ (see (2.15)). A canonical system is definite if and only if the Hamiltonian is of positive type. Hence $\widetilde{\mathcal{H}}$ is of positive type. Since a gauge transformation preserves the deficiency indices we may apply Proposition 5.12 and reach the conclusion. \square

THEOREM 5.14. *Let $S_+ = S_+(J, 0, \mathcal{H})$ be a canonical system ($B = 0$) with a Hamiltonian \mathcal{H} of positive type on \mathbb{R}_+ . For the equation (5.1) to have maximal formal deficiency indices $\mathcal{N}_{\pm}(\mathcal{S}_+) = n$ it is necessary and sufficient that*

$$\int_0^\infty \text{tr } \mathcal{H}(x) dx < \infty. \quad (5.36)$$

PROOF. *Sufficiency.* The inequality (5.36) is equivalent to (5.35) with $k = n$, hence by Proposition 5.12 $\mathcal{N}_{\pm}(\mathcal{S}_+) = n_{\pm}(S_+) \geq n$. On other hand $n \geq \mathcal{N}_{\pm}(\mathcal{S}_+)$ and thus $\mathcal{N}_{\pm}(\mathcal{S}_+) = N_{\pm}(S_+) = n$.

Necessity. Assume that $\mathcal{N}_{\pm}(\mathcal{S}_+) = n$. By Proposition 2.26 (1) also $N_{\pm}(S_+) = n$ and in particular S_+ admits self-adjoint extensions. Fix one of them, say $\widetilde{S}_+ = \widetilde{S}_+^* \supset S_+$.

It follows from Proposition 2.26 (6) that there exists a linear relation $\widetilde{\mathcal{S}}_+$ in $\mathcal{L}_{\mathcal{H}}^2(\mathbb{R}_+)$, satisfying $\mathcal{S}_+ \subset \widetilde{\mathcal{S}}_+ \subset \mathcal{S}_+^*$ and such that $(\pi \oplus \pi)\widetilde{\mathcal{S}}_+ = \widetilde{S}_+$. To calculate the resolvent $(\widetilde{S}_+ - \lambda)^{-1}$ we have to find the solution $\{\widetilde{f}, \widetilde{g}\} \in \widetilde{S}_+$ of the equation $\widetilde{g} - \lambda\widetilde{f} = \widetilde{\psi}$ for an arbitrary $\widetilde{\psi} \in \mathcal{L}_{\mathcal{H}}^2(\mathbb{R}_+)$, or what is the same, the solution $\{f, g\} \in \widetilde{\mathcal{S}}_+$ of the equation $Jf' - \lambda\mathcal{H}f = \mathcal{H}\psi$ with f satisfying some (self-adjoint) boundary conditions at zero and at infinity. It is well-known (see [2], [18]) that

$$f(x, \lambda) = - \int_0^\infty K(x, t, \lambda) \mathcal{H}(t) \psi(t) dt =: K_\lambda(\psi), \quad (5.37)$$

where

$$K(x, t, \lambda) = Y(x, \lambda)[F(\lambda) + 1_{\mathbb{R}_+}(x - t)J^{-1}]Y(t, \bar{\lambda})^*. \quad (5.38)$$

Here $Y(x, \lambda)$ is the fundamental $n \times n$ matrix solution of (5.1) (with $B = 0$) satisfying the initial condition $Y(0, \lambda) = I_n$ and $F(\lambda)$ is some function.

It follows from (5.37) that $K_\lambda(\psi)$ does not depend on the representative $\psi \in \mathcal{L}_{\mathcal{H}}^2(\mathbb{R}_+)$ of $\tilde{\psi}$. Thus K_λ is well defined on $L_{\mathcal{H}}^2(\mathbb{R}_+)$ and in view of (5.37)

$$\tilde{f} = (\tilde{S}_+ - \lambda)^{-1} \tilde{\psi} = \pi K_\lambda(\psi), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \quad (5.39)$$

Combining (5.37)–(5.39) and $\mathcal{N}_\pm(\mathcal{S}_+) = n$ we see that the resolvent $(\tilde{S}_+ - \lambda)^{-1}$ is a Hilbert-Schmidt operator for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Consequently the spectrum $\sigma(\tilde{S}_+)$ is discrete.

Since $S_{\max,+}/S_+$ is finite-dimensional the existence of a self-adjoint extension of S_+ with compact resolvent implies that $S_+ - \lambda$ is a Fredholm relation of index n for all $\lambda \in \mathbb{C}$. On the other hand by Proposition 2.12 (1) we have $\ker(S_+ - aI) = \{0\}$ for all $a \in \mathbb{R}$. Therefore $\dim \ker(S_{\max,+} - aI) = n$. In particular $\dim \ker S_{\max,+} = n$ and by Proposition 2.26 (1) we obtain $\dim \mathcal{E}_0(\mathcal{S}_+) = \dim E_0(S_+) = n$.

But since the system is canonical we have $E_0(\mathcal{S}_+) = \text{span}\{u_j\}_1^n$ with the constant vectors $u_j = \{\delta_{pj}\}_{p=1}^n$. Thus $u_j \in \mathcal{L}_{\mathcal{H}}^2(\mathbb{R}_+)$ for $1 \leq j \leq n$. This is equivalent to $h_{jj} \in L^1(\mathbb{R}_+)$, $1 \leq j \leq n$, that is to the inequality (5.36). \square

To present the next result we recall the following definition.

DEFINITION 5.15. A symmetric system (5.1) is said to be quasi-regular if $\dim \mathcal{E}_\lambda(\mathcal{S}_+) = n$ for all $\lambda \in \mathbb{C}$, that is $\mathcal{N}_\pm(\mathcal{S}_+) = \dim \mathcal{E}_a(\mathcal{S}_+) = n$ for all $a \in \mathbb{R}$.

The following result is a refinement of Theorem 5.14.

THEOREM 5.16. *Under the conditions of Theorem 5.14 the system S_+ is quasi-regular on \mathbb{R}_+ if and only if $\int_{\mathbb{R}_+} \text{tr } \mathcal{H}(x) dx < \infty$.*

PROOF. It is clear that $\mathcal{N}_\pm(\mathcal{S}_+) = n$ if the system S_+ is quasi-regular. Conversely, if $\mathcal{N}_\pm(\mathcal{S}_+) = n$ then the relations $\dim \mathcal{E}_a(\mathcal{S}_+) = n$ for $a \in \mathbb{R}$, have been established in the proof of Theorem 5.14. \square

The next Corollary is derived from Theorem 5.16 exactly as Corollary 5.13 is derived from Proposition 5.12.

COROLLARY 5.17. *Let $S_+ = S_+(J, B, \mathcal{H})$ be definite on \mathbb{R}_+ and $\tilde{\mathcal{H}}$ be as in Corollary 5.13. Then for the system S_+ to be quasi-regular it is necessary and sufficient that*

$$\int_a^\infty \text{tr } \tilde{\mathcal{H}}(x) dx < \infty \quad (5.40)$$

COROLLARY 5.18. *Let $S_+ = S_+(J, B, \mathcal{H})$ be a definite system on \mathbb{R}_+ with constant $J = J(0)$ and such that $\int_1^\infty x \|B(x)\| dx < \infty$. Then for the system S_+ to be quasi-regular it is necessary and sufficient that the condition (5.36) to be satisfied.*

PROOF. It follows from the assumption $\int_1^\infty x \|B(x)\| dx < \infty$ that there exists a fundamental $n \times n$ matrix solution $U(x)$ of the homogeneous equation $JU'(x) + B(x)U(x) = 0$ satisfying

$$U(x) = I_n + o_n(1), \quad x \rightarrow \infty \quad (5.41)$$

where $0_n(1)$ is $n \times n$ matrix function with entries $o(1)$. This fact is well known and can be easily checked (compare with the proof of Proposition 5.37). By Corollary 5.17 S_+ is quasiregular iff $\int_0^\infty \text{tr}(U^*(x)\mathcal{H}(x)U(x))dx < \infty$. In view of (5.41) the last inequality is equivalent to the inequality (5.36). \square

Another criterion for the formal deficiency indices \mathcal{N}_\pm to attain their maximum values n simultaneously (and thus a criterion for the system (5.1) to be quasi-regular) has been obtained in [18]:

PROPOSITION 5.19. [18, Theorem 3.1] *The system (5.1) is quasi-regular on \mathbb{R}_+ if and only if $\dim \mathcal{E}_{\lambda_0}(\mathcal{S}_+) = n$ for some $\lambda_0 \in \mathbb{C}$ and*

$$\inf_{0 \leq t < \infty} \left\{ \text{sgn}(\text{Im}(\lambda_0)) \int_0^t \text{tr}(iJ(t)^{-1}\mathcal{H}(t))dt \right\} > -\infty. \quad (5.42)$$

REMARK 5.20. 1. We emphasize that Theorem 5.14 as well as the other results of this subsection do not depend on J .

2. For Theorem 5.14 (as well as for Proposition 5.12) to hold it is essential that \mathcal{H} is of positive type. Otherwise counterexamples are easy to find.

5.4. Intermediate case.

DEFINITION 5.21. Let A be a linear relation in a Hilbert space \mathfrak{H} and let j be an involution (that is an anti-linear bijective map) in \mathfrak{H} . We will say that A is invariant under j if $\{f, g\} \in A$ implies $\{jf, jg\} \in A$.

LEMMA 5.22. *Suppose that the symmetric linear relation A in \mathfrak{H} is invariant under an involution j . Then $n_+(A) = n_-(A)$.*

PROOF. If $\{f, if\} \in \hat{E}_i(A)$ then $(j \oplus j)\{f, if\} = \{jf, -ijf\} \in A^*$, hence $\{jf, -ijf\} \in \hat{E}_{-i}(A)$. Applying the same argument to j^{-1} one sees that j is an isomorphism from $\hat{E}_\pm(A)$ onto $\hat{E}_\mp(A)$. \square

PROPOSITION 5.23. *Assume that $S_+ = S_+(J, B, \mathcal{H})$ is definite on \mathbb{R}_+ . If both $J^{-1}B$ and $J^{-1}\mathcal{H}$ are real (that is have real entries) then*

- (1) $N_+(S_+) = \mathcal{N}_+(\mathcal{S}_+) = N_-(S_+) = \mathcal{N}_-(\mathcal{S}_+)$;
- (2) if $\dim \mathcal{E}_{\lambda_0}(\mathcal{S}_+) = n$ for some $\lambda_0 \in \mathbb{C}$ then

$$N_\pm(S_+) = \mathcal{N}_\pm(\mathcal{S}_+) = \dim \mathcal{E}_a(\mathcal{S}_+) = \dim E_a(S_+) = n \quad \text{for any } a \in \mathbb{R}. \quad (5.43)$$

PROOF. (1) \mathcal{S}_+ is invariant under complex conjugation and therefore so is S_+ . By Lemma 5.22 $N_+(S_+) = N_-(S_-)$. The other equalities follow from Proposition 2.26 (1).

(2) If $\lambda_0 \in \mathbb{R}$ then the relations (5.43) are implied by Proposition 2.26 (7). If $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ then by (1) $N_\pm(S_+) = \mathcal{N}_\pm(\mathcal{S}_+) = n$. The equality $\dim E_a(S_+) = n$ has been established in the proof of Theorem 5.14 (see also Theorem 5.16). \square

REMARK 5.24. 1. If A is an operator then Definition 5.21 means that A commutes with j . In this case Lemma 5.22 is well-known.

2. The last three equalities in (5.43) meaning the quasi-regularity of the system (5.1) have been established in [2, Theorem 9.11.2] by an analytic method. A generalization of this result is contained in Proposition 5.19. Note however that the condition (5.42), meaning that the formal deficiency indices $\mathcal{N}_\pm(\mathcal{S}_+)$ attain their maximum value simultaneously, does not imply the equality $\mathcal{N}_+(\mathcal{S}_+) = \mathcal{N}_-(\mathcal{S}_-)$ (see Example 5.32 below).

Now we are ready to present conditions for the canonical system (5.1) to have the formal deficiency indices $\mathcal{N}_\pm(\mathcal{S}_+) = n - 1$.

PROPOSITION 5.25. *Let S_+ be a canonical system on \mathbb{R}_+ with a Hamiltonian \mathcal{H} of positive type satisfying*

$$\int_{\mathbb{R}_+} h_{nn}(t) dt = \infty, \quad h_{jj} \in L^1(\mathbb{R}_+), \quad j = 1, \dots, n-1. \quad (5.44)$$

If in addition

$$\left| \int_{\mathbb{R}_+} \operatorname{tr}(iJ^{-1}\mathcal{H}(t)) dt \right| < \infty \quad (5.45)$$

then $N_\pm(S_+) = \mathcal{N}_\pm(\mathcal{S}_+) = n - 1$.

PROOF. Since $h_{jj} \in L^1(\mathbb{R}_+)$, $j = 1, \dots, n-1$ then by Proposition 5.12 $n-1 \leq N_\pm(S_+) \leq n$. Applying Theorem 5.14 we are, in view of condition (5.44), left with three possibilities:

$$N_\pm := N_\pm(S_+) = n-1, \quad (N_+, N_-) = (n-1, n), \quad (N_+, N_-) = (n, n-1). \quad (5.46)$$

We rule out $N_- = n$ and $N_+ = n$. The condition (5.45) yields (5.42) with $\lambda_0 = -i$ and $\lambda_0 = i$. So if $N_- = n$ or $N_+ = n$ then by Proposition 5.19 the system (5.1) is quasi-regular, hence $N_- = N_+ = n$. This contradicts (5.46). Thus $N_\pm = n-1$. \square

COROLLARY 5.26. *Let S_+ be a canonical system on \mathbb{R}_+ with a Hamiltonian \mathcal{H} of positive type such that $J^{-1}\mathcal{H}$ is real. If the condition (5.44) is satisfied then $N_\pm(S_+) = \mathcal{N}_\pm(\mathcal{S}_+) = n-1$.*

PROOF. We show that the condition (5.45) is satisfied and apply Proposition 5.25. Since $J^{-1}\mathcal{H}$ is real so is $a := \operatorname{tr} J^{-1}\mathcal{H}$. On the other hand $a = \operatorname{tr}(J^{-1}\mathcal{H}) = \operatorname{tr}(\mathcal{H}^{1/2}J^{-1}\mathcal{H}^{1/2}) \in i\mathbb{R}$ since J^{-1} is skew-adjoint. Thus $a = 0$. \square

In view of the importance of Hamiltonian systems we reformulate Proposition 5.25 for such systems.

COROLLARY 5.27. *Let $n = 2m$, $B = 0$, $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ and let $\mathcal{H} = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix}$ be the block-matrix representation of a positive type Hamiltonian \mathcal{H} with respect to the*

decomposition $\mathbb{C}^n = \mathbb{C}^m \oplus \mathbb{C}^m$. Suppose that the condition (5.44) holds and that

$$\left| \int_{\mathbb{R}_+} \operatorname{tr}(C_I(t)) dt \right| < \infty, \quad (C_I := (C - C^*)/2i). \quad (5.47)$$

Then $N_{\pm}(S_+) = \mathcal{N}_{\pm}(\mathcal{S}_+) = n - 1$.

COROLLARY 5.28. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathcal{H} = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ be a 2×2 Hamiltonian satisfying $\left| \int_{\mathbb{R}_+} b_I(t) dt \right| < \infty$. Moreover assume that the system S_+ is definite and $\int_1^\infty x \|B(x)\| dx < \infty$. Consider the symmetric extensions of S_+ defined by

$$\tilde{S}_i := \{ \{ \tilde{f}, \tilde{g} \} \in S_{\max,+} \mid f = \operatorname{col}(f_1, f_2) \in \mathcal{D}(\mathcal{S}_{\max,+}), f_i(0) = 0 \}, \quad (i = 1, 2). \quad (5.48)$$

Then

- (1) $N_{\pm}(\tilde{S}_i) = \mathcal{N}_{\pm}(\tilde{\mathcal{S}}_i) = 1$ if and only if $\int_{\mathbb{R}_+} \operatorname{tr} \mathcal{H}(x) dx < \infty$.
- (2) \tilde{S}_i is self-adjoint, i.e. $\mathcal{N}_{\pm}(\tilde{\mathcal{S}}_i) = N_{\pm}(\tilde{S}_i) = 0$ if and only if $\int_{\mathbb{R}_+} \operatorname{tr} \mathcal{H}(x) dx = \infty$.

PROOF. Since the system S_+ is definite then $\mathcal{N}_{\pm}(\tilde{\mathcal{S}}_i) = N_{\pm}(\tilde{S}_i)$. It follows from Proposition 2.26 (3) and (5.48) that $\dim(\tilde{\mathcal{S}}_i/\mathcal{S}_+) = 1$. By Proposition 2.26 (6) we have $\dim(\tilde{S}_i/S_+) = 1$, too. Hence $N_{\pm}(\tilde{S}_i) = N_{\pm}(S_+) - 1 \leq 1$.

1. Let $\int_{\mathbb{R}_+} \mathcal{H}(x) dx < \infty$. Then by Corollary 5.18 $N_{\pm}(S_+) = 2$. and thus $N_{\pm}(\tilde{S}_i) = 1$.

2. Conversely, assume that $N_+(\tilde{S}_i) = 1$ or $N_-(\tilde{S}_i) = 1$. Then $N_+(S_+) = 2$ or $N_-(S_+) = 2$. As in the proof of Proposition 5.25 one now concludes that the system is quasi-regular and hence $N_+(S_+) = N_-(S_+) = 2$. \square

REMARK 5.29. Corollary 5.28 slightly improves a result due to Kac–Krein [16] and coincides with it if $B = 0$ and $b = \bar{b}$, that is $b_I = 0$. Our Theorem 5.14 has been inspired by this result.

Note also that the equalities $N_{\pm}(S_+) = 1$ for 2×2 definite systems with real trace-normed Hamiltonian ($\operatorname{tr} \mathcal{H}(x) = 1$ for $x \in \mathbb{R}_+$) has been established by de Branges [7]. Another proof of the de Branges result has been proposed in the recent publication [13]. These authors have also established an interesting inequality:

$$\langle f(x) - f(y), f(x) - f(y) \rangle_{\mathbb{C}^2} \leq \sqrt{6} |\lambda| \sqrt{|x - y|} \cdot \|f\|_{\mathcal{H}} \quad \text{for } f \in E_{\lambda}(S_+).$$

Now we present some examples clarifying the sharpness of the conditions (5.44) and (5.45) in Proposition 5.25.

EXAMPLE 5.30. Let $J = \operatorname{diag}(i, -i)$, $\mathcal{H} = \operatorname{diag}(h_{11}, h_{22})$ where $h_{jj}(x) > 0$ for $x \in \mathbb{R}_+$. If $h_{11} \notin L^1(\mathbb{R}_+)$ and $h_{22} \in L^1(\mathbb{R}_+)$ then the condition (5.44) holds but the condition (5.45) fails. It is easily seen that $N_+(S_+) = \mathcal{N}_+(\mathcal{S}_+) = 1$ and $N_-(S_+) = \mathcal{N}_-(\mathcal{S}_+) = 2$. If conversely $h_{11} \in L^1(\mathbb{R}_+)$ and $h_{22} \notin L^1(\mathbb{R})$ then $N_+ = \mathcal{N}_+ = 2$ and $N_- = \mathcal{N}_- = 1$.

This example shows that generally speaking Corollary 5.28 does not occur if the condition (5.45) fails.

EXAMPLE 5.31. 1. Let J and \mathcal{H} be as in the previous example. Suppose that $h_{11}(x) \geq h_{22}(x) > 0$ for $x \in \mathbb{R}_+$, $h_{22} \notin L^1(\mathbb{R}_+)$ and $h_{11} - h_{22} \in L^1(\mathbb{R}_+)$. Then $N_{\pm} = \mathcal{N}_{\pm} = 1$ though the condition (5.44) fails and the condition (5.45) holds.

This example shows that the condition (5.44) is not necessary for the relations $\mathcal{N}_{\pm}(\mathcal{S}_+) = n - 1$ to be valid.

2. If $h_{11} - h_{22} \notin L^1(\mathbb{R}_+)$ (say $h_{11} = 2(1+x)^{-1}$, $h_{22} = (1+x)^{-1}$) then again $N_{\pm} = \mathcal{N}_{\pm} = n - 1 = 1$, but neither condition (5.44) nor condition (5.45) hold.

EXAMPLE 5.32. We put $J = J_1 \oplus J_1 \oplus J_1$, where $J_1 = \text{diag}(i, -i)$, $\mathcal{H} = \text{diag}(h_{11}, \dots, h_{66})$, and $h_{11} = h_{33} = 2^{-1}h_{66} \notin L^1(\mathbb{R}_+)$ and $h_{22} = h_{44} = 2^{-1}h_{55} \in L^1(\mathbb{R}_+)$. It follows from Example 5.30 that $\mathcal{N}_+(\mathcal{S}_+) = 4$ and $\mathcal{N}_-(\mathcal{S}_+) = 5$.

On the other hand $\text{tr}(J^{-1}\mathcal{H}) = 0$ and hence the condition (5.42) holds. This example shows that the condition (5.42) is not sufficient for the system (5.1) to have equal formal deficiency indices.

EXAMPLE 5.33. We put in Corollary 5.28 $b(x) = 0$, $a(x) = (1+x)^{-4}$, $c(x) = 1$. Then by Corollary 5.28 the operator S_i is self-adjoint.

On the other hand the eigenvalues of $\mathcal{H}^{1/2}(x)J\mathcal{H}^{1/2}(x)$ are $\pm i(1+x)^{-2}$. Hence we infer that $c^{-1}(x) = (1+x)^{-2} \in L^1(\mathbb{R}_+)$.

This example shows that the conditions of Theorem 5.2 (Theorem 3.2) are not necessary for S_+ to have minimal deficiency indices (to be self-adjoint).

Moreover, this example (as well as Example 3.7) shows that $S_i = S_{\max, i}$ though $\mathcal{D}(S_{\max})$ is not contained in $\text{AC}_0(\mathbb{R}_+, \mathbb{C}^2)$ (cf. Remark 3.6). Indeed, put

$$f = \text{col}((1+x)^{1/4}, 0), \quad g = \text{col}(0, -\frac{1}{4}(1+x)^{-3/4}) \in \mathcal{L}_{\mathcal{H}}^2(\mathbb{R}_+).$$

Then $\{f, g\} \in \mathcal{S}_{\max}$ and $\langle f(x), f(x) \rangle_{\mathbb{C}^2} = \sqrt{1+x} \rightarrow \infty$ as $x \rightarrow \infty$.

5.5. Two-terms Sturm-Liouville equation. Let us consider the equation (2.16) with $Q = R = 0$, that is

$$Py := -\frac{d}{dx}\left(A^{-1}\frac{dy}{dx}\right) = \lambda\mathcal{H}y. \quad (5.49)$$

PROPOSITION 5.34. *Let $A(x)$ be positive definite for all $x \in \mathbb{R}_+$ and $\mathcal{H}(x) \geq 0$ and let $\mathcal{H}(x)$ be a nonsingular on a subset of positive Lebesgue measure. Then for the equation (5.49) to have maximal formal deficiency indices $\mathcal{N}_{\pm}(P_+) = 2n$ (as well as to be quasiregular), it is necessary and sufficient that*

$$\int_0^\infty \text{tr}(\tilde{A}(x)\mathcal{H}(x)\tilde{A}(x))dx < \infty \quad \text{and} \quad \int_0^\infty \text{tr}(\mathcal{H}(x))dx < \infty, \quad (5.50)$$

where $\tilde{A}(x) := \int_0^x A(t)dt$.

If A is uniformly definite on \mathbb{R}_+ , that is $A(x) \geq \varepsilon \cdot I$ ($x \in \mathbb{R}_+$) with some $\varepsilon > 0$ then the second condition in (5.50) is obsolete.

PROOF. As explained in Example 2.5 the system P is unitarily equivalent to a first order system $S(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$, with $\tilde{J}, \tilde{B}, \tilde{\mathcal{H}}$ defined in (2.18). By Proposition 2.18 the system $S(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$ is definite. Then the gauge transformation $Y = \begin{pmatrix} I & -i\tilde{A} \\ 0 & I \end{pmatrix}$ transforms the system $S(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$ into a canonical (and definite) one $S(\tilde{J}, 0, \tilde{\mathcal{H}}_1)$ with \tilde{J} and $\tilde{\mathcal{H}}_1$ defined by

$$\tilde{J} = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{H}}_1 = Y^* \tilde{H} Y = \begin{pmatrix} \mathcal{H} & -i\mathcal{H}\tilde{A} \\ i\tilde{A}\mathcal{H} & \tilde{A}\mathcal{H}\tilde{A} \end{pmatrix}. \quad (5.51)$$

Since the Hamiltonian $\tilde{\mathcal{H}}_1$ is of positive type the first assertion follows from Theorem 5.14.

To prove the second assertion we put $\mathcal{H}_1 := \tilde{A}\mathcal{H}\tilde{A}$ and $\mathcal{H}_2 := \mathcal{H}^{1/2}\tilde{A}^2\mathcal{H}^{1/2}$. Since $A(x) > \varepsilon \cdot I$ one gets $\mathcal{H}_2(x) \geq (\varepsilon x)^2 \mathcal{H}(x)$. Using this and the equality $\text{tr } \mathcal{H}_1(x) = \text{tr } \mathcal{H}_2(x)$ we get

$$\int_1^\infty \text{tr } \mathcal{H}_1(x) dx = \int_1^\infty \text{tr } \mathcal{H}_2(x) dx \geq \varepsilon^2 \int_1^\infty x^2 \text{tr } \mathcal{H}(x) dx \geq \varepsilon^2 \int_1^\infty \text{tr } \mathcal{H}(x) dx.$$

This proves the last statement. \square

Similarly, starting with Proposition 5.12 and taking (5.2a) into account one arrives at the following

PROPOSITION 5.35. *Assume that the conditions of Proposition 5.34 are fulfilled and $\mathcal{H} =: (h_{ij})_{i,j=1}^n$ and $\tilde{A}\mathcal{H}\tilde{A} = (\tilde{h}_{i,j})_{i,j=1}^n$. If*

$$\int_0^\infty h_{jj}(x) dx < \infty, \quad j \in \{1, \dots, k_1\} \quad \text{and} \quad \int_0^\infty \tilde{h}_{ii}(x) dx < \infty, \quad i \in \{1, \dots, k_2\} \quad (5.52)$$

then $\mathcal{N}_\pm(P_+) \geq \max\{n, k_1 + k_2\}$.

COROLLARY 5.36. *Let $0 < c_1 \leq A(x) \leq c_2$ for $x \in \mathbb{R}_+$ and let $\mathcal{H}(x)$ be positive definite on a subset of positive Lebesgue measure. Then for the equation (5.49) to have maximal formal deficiency indices $\mathcal{N}_\pm(P_+) = 2n$ it is necessary and sufficient that*

$$\int_0^\infty x^2 \text{tr } \mathcal{H}(x) dx < \infty.$$

Next we slightly generalize Proposition 5.34. Consider the matrix equation (2.16) with $Q = 0$, that is

$$Py := -\frac{d}{dx} \left(A^{-1} \frac{dy}{dx} \right) + R(x)y = \lambda \mathcal{H}y. \quad (5.53)$$

PROPOSITION 5.37. *Assume that $\mathcal{H}(x)$ is positive definite on a subset of positive Lebesgue measure and*

$$\int_1^\infty \|\tilde{A}(x)\| \cdot \|R(x)\| dx < \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} A(x) \int_x^\infty R(t) dt = 0. \quad (5.54)$$

Then for the equation (5.53) to have maximal formal deficiency indices $\mathcal{N}_\pm(P_+) = 2n$ (as well as to be quasiregular) it is necessary and sufficient that the conditions (5.50) be satisfied.

PROOF. At first we prove that the homogeneous equation (5.53) (with $\lambda = 0$) has two $n \times n$ matrix solutions U and V satisfying:

$$U(x) = I_n + 0_n(1), \quad U'(x) = 0_n(1), \quad x \rightarrow \infty, \quad (5.55)$$

$$V(x) = \tilde{A}(x) \cdot (I_n + 0_n(1)), \quad V'(x) = A(x) \cdot (I_n + 0_n(1)), \quad x \rightarrow \infty \quad (5.56)$$

where as before $0_n(1)$ stands for the $n \times n$ matrix function with entries $o(1)$ as $x \rightarrow \infty$. Indeed it is clear that each solution U of the integral equation

$$U(x) = I_n + \int_x^\infty A(t) dt \int_t^\infty R(s) U(s) ds \quad (5.57)$$

is also a solution of the equation (5.53) with $\lambda = 0$. Choose N such that

$$\int_N^\infty \|\tilde{A}(s)\| \cdot \|R(s)\| ds < 1/2. \quad (5.58)$$

Further, setting $U_0(x) = I_n$ and

$$U_n(x) = \int_x^\infty A(t) dt \int_t^\infty R(s) U_{n-1}(s) ds = \int_x^\infty [\tilde{A}(s) - \tilde{A}(x)] R(s) U_{n-1}(s) ds \quad (n \geq 1)$$

and using (5.58) and the inequality

$$\|(\tilde{A}(s) - \tilde{A}(x))R(s)U_{n-1}(s)\| \leq \|\tilde{A}(s)\| \cdot \|R(s)\| \cdot \|U_{n-1}(s)\|, \quad s > x,$$

one easily proves by induction that $\|U_n(x)\| \leq 1/2^n$ for $n \geq 1$. Hence the series $\sum_{n=1}^\infty U_n(x)$ converges uniformly for $x \geq N$ and $\|\sum_{n=1}^\infty U_n(x)\| \leq 1$. Moreover, the matrix function $U(x) := I_n + \sum_{n \geq 1} U_n(x)$ defines the unique solution of the equation (5.57) (for $x \geq N$) and satisfies the inequality $\|U(x)\| \leq 2$ for $x \geq N$.

Using this estimate one obtains from (5.57) that $U(x) - I_n = 0_n(1)$ as $x \rightarrow \infty$. Differentiating (5.57) and applying (5.54) and the above estimate $\|U(x)\| \leq 2$ one derives the second relation $U'(x) = 0_n(1)$ as $x \rightarrow \infty$.

Thus the existence of the solution U satisfying (5.55) is proved.

To prove the existence of the solution V satisfying (5.56) we recall (see [12], part XI) that for each $n \times n$ matrix solution of the equation (5.53) (with $\lambda = 0$) the matrix function

$$K := U^*(x)A^{-1}(x)U'(x) - (A^{-1}(x)U'(x))^*U(x) \quad (5.59)$$

is constant. Turn x to $+\infty$ and taking (5.55) into account one gets that $K = 0$. This means that U is a self-adjoint solution (in the sence of [12], part XI) of the homogeneous equation (5.53) (with $\lambda = 0$).

Using (5.59) (with $K = 0$) it is easy to check (and it is known (see [12]), that the classical Liouville formula remains valid for the matrix case, that is

$$V(x) := U(x) \int_0^x U^{-1}(t) A(t) (U^{-1})^*(t) dt \quad (5.60)$$

is also a $n \times n$ matrix solution of the equation (5.53) (with $\lambda = 0$).

The relations (5.56) are implied now by (5.55) and (5.60).

Further, following the proof of Proposition 5.34 one transforms the system P to a first order system $S(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$ with $\tilde{J}, \tilde{B}, \tilde{\mathcal{H}}$ defined in (2.18). Then the gauge transformation $Y = \begin{pmatrix} U & -iV \\ iA^{-1}U' & A^{-1}V' \end{pmatrix}$ transforms the system $S(\tilde{J}, \tilde{B}, \tilde{\mathcal{H}})$ into a canonical system $S(\tilde{J}_1, 0, \tilde{\mathcal{H}}_1)$ with

$$\tilde{J}_1 = Y^*(0) \tilde{J} Y(0) \quad \text{and} \quad \tilde{\mathcal{H}}_1 = Y^* \tilde{\mathcal{H}} Y = \begin{pmatrix} U^* \mathcal{H} U & -iU^* \mathcal{H} V \\ iV^* \mathcal{H} U & V^* \mathcal{H} V \end{pmatrix} \quad (5.61)$$

We note that generally speaking $\tilde{J}_1 \neq \tilde{J}$ since $Y(0) \neq I$.

By Theorem 5.14 $N_{\pm}(P_+) = 2n$ iff $\int_1^{\infty} \text{tr}(U^* \mathcal{H} U + V^* \mathcal{H} V) dx < \infty$. By Theorem 5.16 this inequality is also equivalent to the property of the system P_+ to be quasiregular. In view of (5.55) and (5.56) this inequality is equivalent to (5.50)). \square

REMARK 5.38. 1. If $\|A(x)\|$ is bounded ($\|A(x)\| \leq C$), then both conditions (5.54) are implied by the condition $\int_1^{\infty} x \|R(x)\| dx < \infty$.

2. In the scalar case ($n = 1$) the second condition in (5.54) may be omitted.

Next we consider the equation (5.53) with $A = I$. For this case we complement Proposition 5.37.

PROPOSITION 5.39. *Let $A = I$ and let $\mathcal{H}(x)$ be nonsingular on a subset of positive Lebesgue measure. Assume also that $R(x) = k^2 \cdot I_n + R_1(x)$ where $\int_0^{\infty} \|R_1(x)\| dx < \infty$. Then for the equation (5.53) to have maximal formal deficiency indices $\mathcal{N}_{\pm}(P_+) = 2n$ (as well as to be quasiregular) it is necessary and sufficient that:*

$$\begin{aligned} i) \quad & \int_0^{\infty} \text{tr} \mathcal{H}(x) dx < \infty \quad \text{if} \quad k = is \in i\mathbb{R} \quad (k \neq 0); \\ ii) \quad & \int_0^{\infty} e^{2kx} \text{tr} \mathcal{H}(x) dx < \infty \quad \text{if} \quad k > 0. \end{aligned} \quad (5.62)$$

PROOF. i) If $\|R_1\| \in L^1(\mathbb{R}_+)$ then, as it is well known, there exist two $n \times n$ matrix solutions U and V of the homogeneous equation $-y'' - s^2 y + R_1(x)y = 0$ satisfying

$$\begin{aligned} U(x) &= \cos sx \cdot (I_n + 0_n(1)), \quad U'(x) = -s \sin sx \cdot (I_n + 0_n(1)), \quad x \rightarrow \infty, \\ V(x) &= \frac{\sin sx}{s} \cdot (I_n + 0_n(1)), \quad V'(x) = \cos sx \cdot (I_n + 0_n(1)), \quad x \rightarrow \infty. \end{aligned} \quad (5.63)$$

Following the proof of Proposition 5.37 and using the gauge transformation $Y = \begin{pmatrix} U & -iV \\ iU' & V' \end{pmatrix}$ we reduce the equation (5.53) to a canonical system $S(\tilde{J}_1, 0, \tilde{\mathcal{H}}_1)$ with \tilde{J}_1 and $\tilde{\mathcal{H}}_1$ defined in (5.61). In view of (5.63) the inequality $\int_0^\infty \text{tr}(U^* \mathcal{H} U + V^* \mathcal{H} V) dx < \infty$ takes place iff $\int_0^\infty \text{tr} \mathcal{H}(x) dx < \infty$. It remains to apply Theorem 5.14.

ii) Now the homogeneous equation $-y'' + k^2 y + R_1(x)y = 0$ has two $n \times n$ matrix solutions satisfying

$$\begin{aligned} U(x) &= \cosh kx \cdot (I_n + 0_n(1)), & U'(x) &= k \cdot \sinh kx \cdot (I_n + 0_n(1)), & x \rightarrow \infty, \\ V(x) &= k^{-1} \sinh kx \cdot (I_n + 0_n(1)), & V'(x) &= \cosh kx \cdot (I_n + 0_n(1)), & x \rightarrow \infty. \end{aligned} \quad (5.64)$$

Starting with these solutions one completes the proof in just the same way as in the case i). \square

Next we present few results on intermediate formal deficiency indices $\mathcal{N}_\pm(P_+)$.

PROPOSITION 5.40. *Let $\mathcal{H} =: (h_{ij})_{i,j=1}^n$ and $\tilde{A}\mathcal{H}\tilde{A} =: (\tilde{h}_{ij})_{i,j=1}^n$. Assume also that all the functions $\{h_{jj}, \tilde{h}_{jj}\}_1^n$ but one belong to the space $L^1(\mathbb{R}_+)$. Then the formal deficiency indices of the equation (5.49) are $\mathcal{N}_\pm(P_+) = 2n - 1$.*

PROOF. As in the proof of Proposition 5.34 we transform the equation (5.49) to a canonical system $S(\tilde{J}, 0, \tilde{\mathcal{H}}_1)$ with \tilde{J} and $\tilde{\mathcal{H}}_1$ defined in (5.51). One checks that $\text{tr}(\tilde{J}^{-1} \tilde{\mathcal{H}}_1(t)) = 0$. To complete the proof it remains to apply Proposition 5.25. \square

Similarly one proves the following

PROPOSITION 5.41. *Let A , \mathcal{H} and R be as in Proposition 5.37. Then under the conditions of Proposition 5.40 the formal deficiency indices of the equation (5.53) satisfy the inequality $\mathcal{N}_\pm(P_+) \leq 2n - 1$.*

COROLLARY 5.42. *Let $0 < c_1 \leq A(x) \leq c_2$ and $\int_1^\infty x^2 \|R(x)\| dx < \infty$. If*

$$\int_0^\infty (\text{tr} \mathcal{H}(x))^{1/2} dx = \infty, \quad (5.65)$$

then $\mathcal{N}_\pm(P_+) \leq 2n - 1$.

PROOF. Applying the Cauchy-Bunyakovskii inequality one gets

$$\int_1^\infty (\text{tr} \mathcal{H}(x))^{1/2} dx = \int_1^\infty (x^2 \text{tr} \mathcal{H}(x))^{1/2} \cdot \frac{1}{x} dx \leq \int_1^\infty x^2 \text{tr} \mathcal{H}(x) dx. \quad (5.66)$$

Combining (5.65) with (5.66) and taking the obvious inequality $c_1 x \cdot I_n \leq \tilde{A}(x) \leq c_2 x \cdot I_n$ into account one gets $\int_a^\infty \text{tr}(\tilde{A}(x) H(x) \tilde{A}(x)) dx = \infty$

To complete the proof it remains to apply Proposition 5.37 and note that $\mathcal{N}_+(P_+) = \mathcal{N}_-(P_+)$ (see the proof of Proposition 5.40). \square

COROLLARY 5.43. *Consider the scalar ($n = 1$) equation (5.53). Let A and R satisfy the conditions (5.54). Then:*

- (1) $\mathcal{N}_{\pm}(P_+) = 1$ if and only if $\int_0^{\infty} (\tilde{A}^2(x) + 1) \mathcal{H}(x) dx = \infty$.
 (2) $\mathcal{N}_{\pm}(P_+) = 2$ if and only if $\int_0^{\infty} (\tilde{A}^2(x) + 1) \mathcal{H}(x) dx < \infty$.

PROOF. 1) By (5.2a) $\mathcal{N}_{\pm}(P_+) \geq 1$. On the other hand by Proposition 5.37 either $\mathcal{N}_+(P_+) < 2$ or $\mathcal{N}_-(P_+) < 2$. Since maximum values of the formal deficiency indices are attained only simultaneously, one gets $\mathcal{N}_{\pm}(P_+) = 1$.

2) This assertion is a special case of Proposition 5.37. \square

REMARK 5.44. Consider the scalar equation (5.53). In [19] (see also [16]) M. Krein stated (without proof) the following result:

if $n = 1$, $A = 1$, R is semibounded below and $\int_1^{\infty} \mathcal{H}(x)^{1/2} dx = \infty$ then $\mathcal{N}_{\pm}(P_+) = 1$.

It follows from Proposition 5.39 that this result fails. Moreover, we have explicit counterexamples:

$$R = -k^2 < 0, \quad \mathcal{H}(x) = (1+x)^{-2+\varepsilon} \quad (0 \leq \varepsilon < 1).$$

In this case by Proposition 5.39 $\mathcal{N}_{\pm}(P_+) = 2$, but $\int_1^{\infty} \sqrt{\mathcal{H}(x)} dx = \infty$.

Nevertheless Krein's result remains valid for $R \geq 0$ (see Remark 5.11). We emphasize however that the statements of Propositions 5.34, 5.37 and 5.39 are stronger than the statements we obtain by applying Theorem 5.6 to (5.49) and (5.53) respectively. In particular, for $n = 1$ these statements are stronger than Krein's result. Say, if in Corollary 5.43 $A = 1$, $\mathcal{H}(x) = (1+x)^{-3}$ then $\int_1^{\infty} x^2 \mathcal{H}(x) dx = \infty$ and $\mathcal{N}_{\pm}(P_+) = 1$, but $\int_1^{\infty} \mathcal{H}(x)^{1/2} dx < \infty$.

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